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Mechanism Choice and Strategic Bidding in
Divisible Good Auctions: An Empirical Analysis
Of the Turkish Treasury Auction Market

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An Empirical Analysis of the Turkish Treasury Auction Market

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Abstract

An important question in the auctions of divisible goods such as securities, emissions permits, and electricity is to determine whether a discriminatory auction yields higher revenues to the auctioneer than a uniform price or Vickrey auction. The question can be answered with information about the distribution of bidders’ true marginal valuations for the good being auctioned. Since this information is not revealed directly in auction data due to strategic bidding, I model strategic behavior in a discriminatory auction. Based on this model, I propose an estimation method to reconstruct the distribution of marginal valuations using data on individual bids. I apply the method to data from 3-month Treasury bill auctions conducted by the Turkish Treasury between 1991 and 1993. I reconstruct the perfectly competitive outcome in this market using the estimated marginal valuations of the bidders. I find that the discriminatory auction yielded more revenue to the Turkish Treasury than in the competitive outcome. Since the auctioneer’s revenue in the competitive outcome is an upper bound to revenue from a uniform price or Vickrey auction, I conclude that the Turkish Treasury was using the revenue maximizing mechanism among the alternatives. I also find evidence that bidders’ demands become more inelastic when they face binding liquid asset reserve requirements; which supports the Treasury’s choice of a discriminatory scheme for revenue maximization. I establish that my results are robust to the presence of a common-value component in bidders’ utilities when comparing the revenue performance of the discriminatory and Vickrey mechanisms. I also show that the estimation framework can be generalized to allow for deterministic vs. uncertain supply, asymmetries, and stochastic participation by bidders. The empirical method I develop is flexible and computationally straightforward, with potential applications to many other divisible good auction settings.
1 Introduction

What is the most effective way for a Treasury to sell government securities? Since governments sell about $4 trillion dollars worth of securities every year (Bartolini and Cottarelli (1997)), many economists have tried to answer this question, interpreting “effectiveness” from both the revenue maximization and efficiency standpoints. At least since Friedman (1960), the unanimous suggestion of the profession has been to conduct the sale through an auction, citing the fact that buyers of Treasury securities have private information about the value of the security that the Treasury can not hope to guess accurately. Unfortunately there is much less of a consensus among economists as to what the rules of the auction should be. The choice of auction mechanism in Treasury auctions seems to be one of the biggest puzzles of auction theory.\footnote{See the surveys by Bikhchandani and Huang (1993), Nandi (1997), Chari and Weber (1992) and Malvey and Archibald (1998).} This paper addresses the question of mechanism choice empirically, using data on individual bids.

Despite the lack of consensus among economic theorists, in practice, one mechanism is overwhelmingly favored over others. This is the discriminatory auction, also known as the “pay-as-bid” or “multiple-price” auction, used in 39 out of 42 countries surveyed by Bartolini and Cottarelli (1997). In this auction, bidders may submit multiple price-quantity pairs as their bids. Treasury officials sort the price-quantity offers in descending order of price and determine the market clearing price at which total quantity demanded equals the supply of securities. Price-quantity pairs above the market clearing price win the quantity they specify, and pay the price they indicate for that quantity.

The multiple price-quantity pairs that each bidder specifies can be regarded as tracing out a bid function on the price-quantity plane. The market clearing price is found by aggregating individual bid functions and finding where the aggregate bid function meets supply, as in Figure 1(a). The total revenue that the auctioneer gets from the auction, then, is the area under the aggregate bid function up to the market clearing price.

An alternative mechanism is the uniform price auction. Here, winning bids are determined in the same way.

\textbf{Figure 1:} Discriminatory and uniform price auctions
manner as in the discriminatory auction, but bidders pay the market clearing price for all the units they purchase. In this case, the auctioneer’s revenue is the rectangle defined by the total quantity being sold and the market clearing price, as shown in Figure 1(b).

To understand the revenue tradeoff between these two mechanisms, a brief look at an individual bidder’s problem is in order. In Figure 2, I draw what a bid function looks like for a given bidder. Observe that this is a step a function due to the finite number of price-quantity pairs. I also plot the residual supply function for this bidder, calculated by subtracting the aggregate bid function of all other bidders from the total supply. The market clearing price of the auction is at the point where the individual bid function intersects the residual supply function. This point of intersection also defines the total quantity won by the bidder.

![Figure 2: Bidder’s problem](image)

In the discriminatory auction, the bidder pays the area under his bid function up to the quantity he wins, area $A + B + C$ in Figure 2. Therefore, for any given quantity, a rational bidder would bid a price that is lower than his true (marginal) valuation for that quantity, i.e. “shade” his bid. The amount by which a bidder shades his bid relative to his valuation, however, depends on where he believes the market clearing price, or equivalently, the residual supply function will lie. If competing bidders have private information about their marginal valuations, the residual supply functions for each bidder will be random. Hence, what is relevant for a strategic bidder trying to make an optimal decision under uncertainty, is the distribution of residual supply functions that he is going to face.

The incentive for bid-shading is not as strong in the uniform price auction, since the bidder only pays area $A + B$ in Figure 2, the rectangle defined by the market clearing price and the total quantity he wins. In
fact, as originally discovered by Vickrey (1961), if the auctioneer charges the bidder only the area under the residual supply curve (area $A$), then the bidder’s optimal response will be to bid his marginal valuation.\(^{2}\)

Since bidders do not shade their bids as much, the market clearing price in a uniform price or Vickrey auction will be higher than in a discriminatory auction. Hence, the auctioneer extracts higher revenues from marginal units. However, in the discriminatory price auction, the auctioneer extracts revenue from inframarginal units. Therefore, the revenue tradeoff between a discriminatory vs. a uniform price or Vickrey mechanism depends on the amount of bid-shading each bidder decides to undertake in the discriminatory auction.

This paper studies strategic behavior in a discriminatory auction, building an empirical framework in which the amount of bid-shading undertaken by each bidder can be estimated using individual bidding data. This allows me to calculate the unobserved marginal valuations of the bidders, and to conduct counterfactual calculations in which the revenue performance of a uniform price or Vickrey auction can be compared to the performance of a discriminatory auction. I then apply the framework to data from the Turkish Treasury, covering 3-month T-bill auctions between 1991 and 1993. I find that the discriminatory mechanism yielded higher ex-post revenues to the Turkish Treasury than a uniform price or Vickrey mechanism would have.

Specifically, building on previous work by Wilson (1979), I model bidder behavior in a discriminatory auction as an incomplete information game. Since in reality bidders are confined to submitting a finite number of price-quantity pairs on a finite price-quantity grid, my model also borrows features from the work of Nautz (1995), who models a non-strategic version of the discriminatory auction. The model yields a first-order necessary condition in which the price bid for a given quantity is determined as the marginal value for that quantity less a “shading factor,” which depends on the distribution of residual supply functions that the bidder expects to face.

The methodological contribution of the paper is to develop an estimator of the “shading factor.” Once again, the amount of bid shading undertaken by a particular bidder is determined by the probability distribution of the residual supply functions he expects to face. If bidders have rational expectations, as they would in an equilibrium of the incomplete information game, the residual supply function that the bidder actually faces is a draw from this distribution.

Assuming that private information is distributed independently across bidders and that bidders are symmetric, the equilibrium bid functions (or “bid vectors,” since I focus on the discrete game) are independent, identically distributed random variables. The residual supply function is the sum of these random variables, subtracted from total supply. Hence, different realizations of the residual supply function can be obtained by making random draws from the empirical distribution of bid vectors observed in the data, and calculating the

\(^{2}\)If the bidder submits a bid function above his marginal valuation, then he pays extra for no reason. If he submits a bid function that is below his marginal valuation, he loses out on units he could have won for certain, as the residual supply function is completely independent of his bid function.
residual supply function corresponding to these draws. If there are enough observations of the bid vectors in the data, the empirical distribution closely resembles the true distribution. Hence, the distribution of residual supply functions obtained from draws from the empirical distribution of bid vectors resembles the true distribution of residual supply functions that a bidder conditions his optimal response on.

By recovering the distribution of residual supply functions in this way, I can calculate the shading factor between a bidder’s price offer and his marginal valuation for each quantity. I find that some inframarginal bids can only be rationalized by ascribing very high marginal valuations to them, as the prices submitted by the bidder are almost certain to be above the market clearing price. This points to inelastic bidder demands for a certain fraction of inframarginal units. I find support for the hypothesis that this inelasticity is caused by banking regulations in Turkey: through “liquid asset reserve requirements,” Turkish banks are required to hold at least 30% of their portfolios in Treasury securities. I find evidence that banks who failed to win a large portion of their total demand in the previous auction bid very aggressively in the current auction.

My estimates of bidders’ marginal valuations also allow me to reconstruct the perfectly competitive outcome of this auction, in which each bidder reveals his marginal valuations truthfully. The Treasury’s revenue in a perfectly competitive outcome constitutes an upper bound for the its revenues with a uniform price or Vickrey auction.

In extensions of the empirical method, I introduce a common value component into my model by allowing bidders’ ex-post utilities to depend explicitly on the realized market clearing price of the auction. I show that revenue comparisons based on a private value specification are robust to the addition of this common value component. I also establish the robustness of my results in the presence of supply uncertainty, asymmetries among bidders, and stochastic participation.

Since the original argument of Friedman (1960) in favor of a uniform price auction, economists studying auctions have vigorously debated whether the uniform price auction would yield higher revenues to the Treasury than the discriminatory price auction. Unfortunately, auction theory has not yet been able to resolve this debate. The celebrated “revenue equivalence theorem” of auction theory can not, in general, be applied to compare the discriminatory price and uniform price mechanisms. In the case of common or affiliated values, a theoretical comparison becomes even more difficult. Less general theoretical results using specific functional assumptions and simplified settings do not yield a conclusive answer to the question.

\footnote{Since the realized market clearing price of the auction is a function that aggregates the private information of all bidders, it enters as a common component to bidders’ valuations and causes a “winner’s curse” effect.}

\footnote{For example, see Klemperer (2000) for a prominent auction theorist’s opinion on the current state of affairs.}

\footnote{Recent results by Ausubel and Cramton (1997) suggest that even with independent private values, these mechanisms can lead to inefficient allocations. Since the revenue equivalence theorem requires two compared mechanisms to be result in identical allocations – the efficient allocation being one possibility, – it is not possible to establish the equivalence of expected revenue across two mechanisms which result in possibly very different allocations.}

\footnote{See, for example,Bikchandani and Huang (1989), Back and Zender (1993), Wang and Zender (1995), Noussair (1995).}
Previous empirical research addressing the question of mechanism selection for Treasury auctions has mostly focused on “policy experiments” in which different auction formats have been used in different time periods or in the sale of securities of different maturities. These studies compare the differential between the auction price and the resale or forward (“when-issued”) market price of the security across separate samples of discriminatory and uniform price auctions. Umlauf (1993) utilizes a data set of Mexican Treasury auction that straddles a policy change from the discriminatory mechanism to the uniform price mechanism. Using price data from the resale market, reports statistically significant but small revenue gains by the use of the uniform price mechanism. Simon (1994) reports large losses of revenue from a switch to the uniform price mechanism by analyzing a similar shift undertaken in the 70’s by the U.S. Treasury, using resale data. Using price data from the forward (“when-issued”) market, Nyborg and Sundaresan (1996) and Malvey and Archibald (1998) find positive but statistically insignificant benefits from the use of the uniform price mechanism in a similar experiment conducted by the U.S. Treasury in the 90’s.

The assumption implicit in this research is that bidders’ true valuations for the security are better reflected in resale or forward markets than in the auction. Hence a smaller differential between the auction price and the transaction prices in these markets reflects better surplus extraction by the auctioneer. Another assumption is that bidders’ demand for the securities is perfectly elastic, a point that is contested in Section 6 of this paper, along with several papers in the finance literature. Furthermore, in many circumstances, data from such policy experiments is not available, restricting the applicability of the methodology.

All except one of the policy experiment studies cited above have relied on aggregated price data from the auctions, rather than utilizing bidder level data. An important advantage of this paper is that I have access to individual bid data. This allows me to base my empirical method on a model of individual bidding behavior, as opposed to drawing conclusions from auction aggregates.

Starting with Paarsch (1992), empirical auction literature has developed various methods to compute the unobserved values of the bidders using behavioral models of bidding. The contribution of this paper to the existing empirical literature on Treasury auctions and the empirical literature on auctions in general is that it is the first empirical study whose theoretical foundations are consistent with a *strategic divisible good auction* framework. A recent paper by Heller and Lengwiler (1998) has independently made the observation that the discriminatory vs. uniform price auction choice can be analyzed using optimality conditions governing bidder behavior; however, as they point out, their model and estimation method is not consistent with a strategic equi-

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7 See, for example, Shleifer (1986), and Bagwell (1992).
8 The exception is Umlauf (1993), who nevertheless treats all bidders as having the same value for the security.
9 In the long literature on Treasury auctions, only Umlauf (1993), Gordy (1994), Beyazitoglu and Kiefer (1997), Gordy (1999), and Nyborg, Rydqvist and Sundaresan (1997) report access to such data. In particular, none of the empirical studies on U.S. Treasury auctions that I know of has used bidder level data, except for Scott and Wolf (1979), who use proprietary data from a single bidder.
librium framework.\textsuperscript{10} Previous attempts to analyze Treasury auctions within a strategic equilibrium framework have been constrained to use models of single-unit auctions, since strategic bidding in divisible good auctions is difficult to analyze both analytically and computationally.\textsuperscript{11} The method developed in this paper offers an alternative to the difficult computational problem of characterizing equilibria explicitly within the estimation algorithm. In the spirit of Elyakime, Laffont, Loisel and Vuong (1994), and Guerre, Perrigne and Vuong (2000), I implement an empirical procedure that has low computational demands and minimizes the impact of distributional assumptions on estimation results. A paper that also benefits from the methodological insight of Elyakime et al. (1994), and Guerre et al. (2000), in a strategic context is Gordy (1994), which analyzes the effect of reservation price policy in Portuguese Treasury auctions. However, Gordy’s structural model treats the quantity choice of a bidder as being exogenous, and does not account for multiple price-quantity pairs submitted by a single bidder, a defining feature of Treasury auctions.\textsuperscript{12}

The outline of the paper is as follows: In section 2, I discuss the institutional setup of the Turkish Treasury auction market, and report summary statistics from my data. In section 3, I present a model of strategic bidding in a discriminatory price divisible good auction, which is motivated by the findings in section 2. Section 4 discusses how the model developed in section 3 can be used as an empirical device to estimate the unobserved marginal valuations of the bidders. The simulation study concluding this section highlights the workings of the empirical method. In section 5, I use bidding data from the Turkish Treasury bill auctions between October 1991 and October 1993 to estimate the marginal valuations of the bidders. Using my estimates, I conduct comparisons between the discriminatory, uniform price and Vickrey mechanisms. In section 6, I discuss the robustness of my results in the presence of a common value component, supply uncertainty, asymmetries, and stochastic participation. Section 7 concludes. The appendix contains proofs of the propositions and provides further discussions of the empirical method.

\textsuperscript{10}Observe that the very related idea of using first-order conditions to reconstruct unobserved marginal cost functions from observed supply decisions has existed in the industrial organization literature at least since Rosse (1970) and Bresnahan (1981). Heller and Lengwiler (1998) have access to data from a uniform price auction and have to reconstruct the hypothetical outcome in a discriminatory auction. They model bidders’ expectations about the market clearing price based on the distribution of secondary market prices, but do not consider the strategic effect each bidder has on the distribution of the market clearing price. Hence, their estimates of the “shading factor” do not have an equilibrium foundation.

\textsuperscript{11}Nyborg et al. (1997) motivate their empirical investigation with a model of bidding; however, they use a single-unit sealed-bid first-price auction model, and do not estimate structural parameters of their model. Scott and Wolf (1979) and Beyazitoglu and Kiefer (1997) also follows a model-based approach to analyze bidding behavior. However, their models of bidding, like Heller and Lengwiler (1998), do not allow for strategic interaction among bidders.

\textsuperscript{12}Gordy (1994) lumps together multiple price-quantity bids of an individual bidder into a single composite bid, and treats this as his unit of observation, with price being the sole strategic variable. A subsequent paper, Gordy (1999), argues that multiple bids are a device to overcome the winner’s curse; however, a theoretical model for this explanation is not provided.
2 An Overview Of The Market

This paper uses data from 13-week Treasury bill auctions conducted by the Turkish Treasury between October 1991 and October 1993. The dataset consists of price and quantity pairs for each bidder (coded by an ID number), the quantity of Treasury bills supplied, and the market clearing prices for the auctions.

Due to the inflationary environment in the last decade, the Turkish government has been constrained to borrow in the very short term. The 13 week bills I focus on comprise about three-eighths of the bond issues of the period. Hence, the data set provides a good representation of the workings of the domestic debt market at the time.

The macroeconomic conditions during the sample period, October 1991-October 1993, were quite volatile, but not atypical of the overall macroeconomic environment of the past decade. The average annual inflation rate in this period was 68%. The budget deficit grew from 8% of GDP to 25% of GDP. Although the Central Bank made an effort to follow a well articulated “independent monetary program” to prevent fiscal abuse and the onset of a hyperinflation, there were some rifts with the cabinet towards the end of the sample over the use of Central Bank “advances” to the Treasury. I should also note that a major financial crisis and an ensuing recession occurred in Turkey circa May 1994.

2.1 The auction

The Turkish Treasury has auctioned off government debt since May 1985. Short term (13 week to 52 week) securities offered by the Treasury are pure discount bills and bonds and do not bear coupons. The auctions are held on Wednesdays, rotating through 52, 39, 26 and 13 week bonds. Bidding is open to the general public; however the majority of bids come from banks, brokerages, and other financial institutions. Financial institutions have to put up a collateral equaling 1% of the face value of their bids; the general public is subject to a 100% collateral requirement, causing most private parties to bid through their broker or banker.

The Turkish Treasury uses the discriminatory auction format. Each bidder is asked to specify a price and a quantity demanded at that price. Prices are quoted as the amount a bidder is willing to pay for an imaginary T-bill with face value of 100 Turkish Liras (TL). Prices can be specified up to 3 significant digits. Quantities are specified in terms of the face value of T-bills the bidder wants to buy. The minimum quantity a bidder can request to buy is 50 million TL (about $6000). There is no limit to the number of price-quantity pairs submitted by a bidder. In fact, the average number of price-quantity pairs submitted in my sample is 6.9, with one bidder submitting over 60 price-quantity pairs per auction. Bids are submitted by noon on Wednesdays.

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13 I thank Berna Beyazitoglu for providing the data.
14 A reading of period newspapers reveals that the acrimony escalates in summer 1993, resulting in the replacement of the Central Bank governor.
Auction results are announced at 5:00 PM and winning bids are settled 1 week after the auction.\footnote{There are three main differences between the institutional setup of Turkish Treasury auctions and U.S. Treasury auctions: First, there is no when-issued (forward) trading in the Turkish T-bill market. Second, unlike the U.S. Treasury auctions, non-competitive bids (bids that do not specify a price, and are automatically filled at the prevailing market-clearing price) are not allowed in the Turkish Treasury auctions. Third, bidders in the Turkish T-bill auctions are not subject to maximum quantity constraints. In the U.S. bidders are constrained to at most 35\% of the auction issue.}

In the period I study, October 1991 to October 1993, the Treasury followed two different procedures in the allocation of the bills. Until February 1993, the total quantity of bills sold in the auction was not announced until after the bids were submitted. However, bank managers that I have interviewed claimed they could estimate the supply of Treasury bills quite accurately by tracking debt service requirements of the Treasury, and through contacts in the Treasury.\footnote{I will discuss this point in more detail in Section 6.2.} Beginning February 1993, the Treasury began to pre-announce the quantity in its 3 and 6 month Treasury bill auctions in an attempt to commit to restricting the supply of short term securities and increase the average maturity of outstanding government debt.\footnote{This change in policy is not mentioned in Beyazitoglu and Kiefer (1997), who maintains that the Treasury did not announce the issue volume throughout the period. However, conversations with Treasury officials and period newspapers (for example the February 3rd 1993 issue of Cumhuriyet) confirm the policy shift.}

2.2 The bidders

Banks are the main players in the market, as they capture 93\% of the bills sold in the sample. Brokerages buy 6\%, and other bidders (institutional investors, insurance firms, pension funds) share the remainder. The top 5 bidders capture 30\% about evenly, and top 15 capture about 70\% of the issues. However, the division of market share among these banks does not differ by more than 3\%. This is about the same level of concentration as in the overall banking sector, in which top 15 banks own 65\% of banking assets about evenly.

Participation varies widely among the total 134 participants observed in the auctions in the sample. 48 bidders entered more than 20 of the 27 auctions and 50 entered less than 10 auctions in total.

According to a survey of 51 banks and other financial institutions conducted by Alkan (1991) in late 1989, the main reasons for participating in the Treasury auctions (of all maturities) are as follows:

1. 42\% of total purchases in the auctions are to meet the liquid asset reserve requirements monitored by the Central Bank. In the period I study, at least 30\% of bank portfolios had to be held as government bonds and bills, with an average maturity of 210 days. Failure to comply with this requirement resulted in monetary fines and possibly a suspension of bank operations.

2. 37\% of total purchases are for resale in the secondary market.

3. 10\% of total purchases are to fill customer orders.
4. 10% of purchases are to fulfill collateral requirements, for investment funds administered by the bank, and for buy-and-hold purposes.

According to Alkan (1991), the relative importance of fulfilling reserve requirement vs. resale seems to vary a lot among survey respondents. Alkan (1991) points out that resale incentives might be more pronounced in 13 week T-bill auctions than in auctions of higher maturity securities. However, “buy-and-hold” strategies might also have been an important factor in the period I study, as short-term government securities yielded outstanding returns to investors.

Alkan (1991) also reports that bidders find the following bids of information as being useful in their bidding decision (listed in order of importance): Treasury’s borrowing requirements and repayment schedule, liquidity in money markets, the bidder’s own liquid asset reserve requirement and the reserve positions of other bidders, conversations with other bidders, and results of previous auctions.

Information on the Treasury’s borrowing requirement and repayment schedule can be used in two ways. The first is to obtain a measure for the default risk premium, though in section 2.4, I argue that this might not be big concern. The second is to reduce supply uncertainty. Banks who enter the auction to meet their liquid asset reserve requirements need this information as a critical input for assessing the probability of winning their minimum required quantity. Those who bid for speculative purposes need the information to estimate the resale price of the security following the auction.

Liquidity in money markets is an important determinant of the value of securities being auctioned. In the money market, Turkish banks routinely enter into “repo” and “reverse-repo” agreements with other banks or individuals, where they engage in short-term borrowing (“repo”) or lending (“reverse repo”), using Treasury securities as collateral.

Carrying conversations with other bidders up to minutes before the auction is a shared characteristic of Treasury auction markets around the world. Such conversations can be thought of uninformative “cheap-talk” before the auction. A more serious implication of such conversations is to regard them as an attempt to collude. However, no official complaint has been filed in Turkey against collusion in these auctions. Market participants and Treasury officials that I have been able to talk to do not believe collusion is a problem in this market.

As for the strategic and analytic sophistication of the bidders: since the government securities market is the largest organized financial market in Turkey, bidders expend significant resources to strategize. Personal interviews with managers of two mid-scale private banks have revealed that these banks have developed proprietary analytic and decision support software to aid their bidding decisions.

2.3 Secondary Markets

The main venue for trading in government securities, the Istanbul Stock Exchange Bonds and Bills Market (ISEBBM), was established in June 1991 as a computerized double-auction market. Table 1 reports transaction
### Table 1: Auctioned debt stock vs. secondary market volume

| Year | Auctioned debt
| (billion TL) | ISE total volume
| (billion TL) |
|------|-----------------|------------------|
| 1991 | 11,510          | 1,685            |
| 1992 | 7,497.1         | 2,110.6          |
| 1993 | 17,221.5        | 16,074.9         |

*a Data from the Turkish Ministry of Treasury, covering auctions of all maturities.
*b Data from the ISE webpage: www.ise.org

Volumes in this market in comparison to primary auction market volumes. Using daily transaction data from the ISEBBM, I have calculated that, on average, 4% of the total volume of auctioned Treasury bills are traded on the ISE in a 3-day post-auction window. Based on this figure one could draw the conclusion that liquidity in the resale market is not very large compared to the primary auction market, especially before 1993. This would imply that the case for a common resale value of the auctioned securities is not as strong as it is believed in previous theoretical and empirical literature. I should note, however, that trading volumes on the ISEBBM showed a strong growth trend during the period. Further, ISEBBM was not the only resale market available to the bidders, who also participated in over-the-counter deals. Unfortunately, I do not have access to data on these transactions.

### 2.4 Descriptive statistics

Table 2 displays various summary statistics for the auctions in the data set. Monthly inflation data was obtained from the Central Bank of Turkey. The total number of bidders corresponds to the total number of unique bidders showing up in the data set. Revenues are converted to US dollars using daily exchange rate data. Cover ratio is the ratio of the number of T-bills sold in the auction to the number of T-bills demanded by bidders. The auction yield is the quantity weighted average yield of the T-bills that were sold in the auction. To get an indication as to the amount dispersion of opinion among bidders regarding the auction interest rate, I calculate the quantity-weighted variance of the price bids.

As we can see from Table 2, the ex-post real interest rates realized in the auction are very high (about 23%). The high interest rate burden of the Turkish treasury has been hailed as the “cornerstone of Turkish macroeconomic imbalances” by a recent Goldman Sachs report (Munir (1999)). However, strategic bidding by itself can not account for the government’s high cost of borrowing, as secondary market yields do not reflect a vast “underpricing” in the auctions. While interpreting the real interest rates implied by the above table, one should remember that they are calculated ex-post, and do not take account the risk premium associated
Table 2: Summary statistics for 3-month T-bill auctions

<table>
<thead>
<tr>
<th>Variable</th>
<th>Average</th>
<th>Std.dev.</th>
<th>Min</th>
<th>Max</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation(^a)</td>
<td>67.8%</td>
<td>5.6%</td>
<td>58.0%</td>
<td>78.7%</td>
<td></td>
</tr>
<tr>
<td>No. of bidders</td>
<td>69</td>
<td>21.02</td>
<td>32</td>
<td>110</td>
<td>134(^b)</td>
</tr>
<tr>
<td>Revenue (million $)(^c)</td>
<td>456.3</td>
<td>443.1</td>
<td>31.4</td>
<td>1560.9</td>
<td>12369.1</td>
</tr>
<tr>
<td>Cover ratio(^d)</td>
<td>28.9%</td>
<td>24.9%</td>
<td>0.2%</td>
<td>84.3%</td>
<td></td>
</tr>
<tr>
<td>Auction yield(^e)</td>
<td>90.8%</td>
<td>6.9%</td>
<td>79.0%</td>
<td>101.4%</td>
<td></td>
</tr>
<tr>
<td>Variance of bids(^f)</td>
<td>0.035</td>
<td>0.031</td>
<td>0.005</td>
<td>0.134</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\)Monthly CPI from the Central Bank of Turkey
\(^b\)Total number of unique bidders who participated in the auctions.
\(^c\)Converted using daily exchange rate data
\(^d\)Quantity sold/Quantity demanded
\(^e\)Quantity weighted average yield
\(^f\)Calculated using the formula: \[
\frac{\sum_{i=1}^{N}(p_i - p_{avg})^2 q_i}{\sum_{i=1}^{N} q_i}
\]

with a highly variable inflation rate and a government with a growing debt burden. The default risk premium is probably not high enough to account for the overall premium, considering that the Turkish government was borrowing at an average 9% in dollar and DM dominated debt in European and U.S. markets in this period. The primary explanation for the high real interest rates should be sought in a large inflation/depreciation risk premium, which was partly justified by a devaluation of the Turkish lira in May 1994. In fact, Berument and Malatyali (1999) find empirical support for a significant inflation/devaluation risk premium by estimating a model of the Treasury auction rate that controls for expected inflation risk.

The second observation from Table 2 is that the percentage of demand fulfilled in an auction varies widely. As we noted before, in response to the shortening maturity structure of the debt, the Treasury began a conscious effort to limit the 3-month bill supply beginning in February 1993. I will account for the effect of supply uncertainty in my estimation strategy in section 6.

The average variance in bid prices translates to about 2% in annual yield. This bid spread is very high compared to U.S. standards, where the spread is several basis points (one-hundredth of a percent). Large bid spreads can be interpreted as there being a wide variation in bidder information. Private information can either be about individual liquid asset reserve needs, which can be regarded as a bidder’s private valuation for the T-bills, or about resale prospects, a common value component. The next section provides a model in which bidders receive private information about the value of the Treasury bills and strategize their bids accordingly.
3 A model of bidder behavior

I will model the discriminatory price auction as a game in which bidders possess private information about the value of the security. Wilson (1979) provides the first model of a divisible good auction with this feature.

In Wilson’s model, bidders are allowed to submit continuous bid functions rather than discrete price quantity pairs. In this section, I will assume that bidders are constrained to submit price-quantity bids on a finite grid. Analyzing the discrete case is more realistic for the Turkish Treasury auction market, since there is a smallest price increment (0.001 TL), and a minimum bid quantity restriction. The discreteness assumption transforms the game into a finite game of incomplete information, for which the existence of an equilibrium can be obtained.

A discrete model of bidding in a discriminatory price auction in which bidders are non-strategic is also modeled in Nautz (1995), from whom I borrow certain aspects of my model setup.

Specifically, let there be \( N \) bidders in the auction, where \( N \) is common knowledge. The total supply of the good or security for sale is \( Q \). Assume, for now, that \( Q \) is deterministic; in section 6, I will discuss the case in which \( N \) and \( Q \) can be random.

Let \( p_0 < p_1 < \cdots < p_{K+1} \) denote the set of possible prices on the grid. The bid vector submitted by each bidder is defined as quantities specified for each of these prices: \( \mathbf{y}_i : \{ y_{i0} \geq y_{i1} \geq \cdots \geq y_{iK+1} \} \). Quantities are also constrained to be on a finite grid, i.e. \( y_{ij} \in \{ y_0, \cdots, y_{M+1} \} \), where the grid interval is \( \Delta y \).

Looking back at Figure 2, these quantity bids at different price points make up the “step function” that each bidder submits.

After all bids are submitted, the auctioneer determines the market clearing price by aggregating the quantity bids for each point on the price grid and finding the price at which total demand just falls short of the total supply, i.e.

\[
p_{k^*} : k^* = \min \{ k : \sum_{i=1}^{N} y_{il} \leq Q \}
\]

Observe that defining the market clearing price in this way enables me to avoid technical problems associated with having to ration quantities when demand exceeds supply at the clearing price. This might be regarded as an unnatural assumption in an auction environment where rationing occurs frequently. My conversations with auction insiders, however, indicate that rationing has not been a big concern in the Turkish market.

The information and payoff structure of the game are as follows: let \( \Omega_t \) denote the set of economic variables that are relevant to valuing the securities being auctioned, and that are publicly observable by all bidders preceding auction \( t \). Variables that can enter \( \Omega_t \) are the prices of securities that are close substitutes to the security on auction, latest monetary and fiscal announcements, or general market news. \( \Omega_t \) can be thought of all the information bidders in these auctions can gather from their trading screens and public information sources.
Aside from observing $\Omega_t$, each bidder also receives a private signal, $s_i$. The signal space can be any complete separable metric space for purposes of obtaining existence of an equilibrium; however, for ease of exposition, assume that $s_i$ is a scalar. Assume the private signals $s_i$ are distributed over a compact support with the atomless marginal distribution $F_i$. The economic counterpart of these signals can be banks’ liquid asset reserve requirements, market forecasts obtained from proprietary sources, or observation of customer order flows for the security.

As for the utility specification, assume that bidders are risk-neutral, and have the marginal valuation $v_i(y, s_i, \Omega_t)$ for $y$ units of the security, given the private information in the signal $s_i$ and the public information in $\Omega_t$. Assume that the marginal valuation is decreasing in $y$.^{18}

One of the main assumptions of this paper regards the information structure of the auction:

**Assumption 1.** Conditional on $\Omega_t$, private signals $s_i$ are independent.

The main justification for this assumption is that a lot of information is publicly observable prior to the auction. Bidders in this market are on the phone with each other until minutes before the bids are due, and all serious bidders in this market have access to the same public information sources. Therefore, conditional on all public information, it is reasonable to assume that “differences of opinion” among bidders, modeled by $s_i$, will be independently distributed.^{19}

The fact that I do not allow correlation across private signals does not rule out the presence of a common value component in bidders’ utilities. I analyze a case in which bidders’ values can be linked through a common value in Section 6. However, observe that with the present specification for the valuation function, the independence assumption makes the setting one of independent private values. This means that learning other bidders’ signals will not change the valuation of a given bidder.

As discussed in section 2, one of the most important pieces of private information in this market is a bidder’s liquid asset reserve position. If a bank does not satisfy the liquid asset reserve requirement by winning enough T-bills in the auction, it has to resort to the money market or the secondary market to balance its reserve position, or pay a fine to the government. The expected cost of these alternatives, hence the opportunity cost of not winning a given amount of securities in the auction, will depend on the bank’s current reserve position. In this case, it is not a bad assumption to think that information about another bank’s liquid asset reserve position will not affect the value a bank attaches to the securities.

---

^{18}Risk neutrality vs. risk aversion of banks is an issue widely discussed in the banking literature. Here, I follow the convention of the auction theory literature in assuming that the bidders are risk neutral. One could also think that a downward sloping marginal valuation schedule for the securities implicitly accounts for risk aversion. This argument is not exactly tight, however, since risk averse behavior towards uncertainty in the auction outcome is not accounted for.

^{19}A reason to assume independence of the private signals is to obtain the existence of an equilibrium to the game, as discussed below. Athey (2000) has some recent results regarding existence of an equilibrium with correlated signals in the single unit auction case. The extension of Athey’s results to the multi-unit case has recently been discussed by McAdams (2000).
Since bidders have private information, the market clearing price is a random variable from the perspective of each bidder. Define the probability that the market clearing price is below \( p_k \) conditional on the bid vector of bidder \( i \), \( \overrightarrow{y}_i \), to be:

\[
H(p_k, \overrightarrow{y}_i) = \Pr\{\text{market clearing price} \leq p_k, \text{ given } \overrightarrow{y}_i\} \tag{2}
\]

I will make use of this expression frequently.\(^{20} \)

From now on I will assume, without loss of generality, that \( H(p_0, \overrightarrow{y}_i) = 0 \), that is, the price grid is chosen large enough that no matter what the bidders do, the market clearing price will remain above the lowest point on the grid, \( p_0 \).

Given this setup, the expected payoff of a risk neutral bidder who submits the bid vector \( \overrightarrow{y}_i = \{y_{i0}, y_{i1}, \ldots, y_{iK}, y_{iK+1}\} \) will be:

\[
\sum_{k=1}^{K} [\Pr\{p_{k-1} \leq \text{market clearing price} \leq p_k, \text{ given } \overrightarrow{y}_i\}] \times \{\text{Profit made on bids above } p_{k-1}\} = \sum_{k=1}^{K} [H(p_k, \overrightarrow{y}_i) - H(p_{k-1}, \overrightarrow{y}_i)] \times \left\{ \sum_{j=k}^{K} \left( \int_{y_{ij+1}}^{y_{ij}} v_i(q, s_i) dq - p_j(y_{ij} - y_{ij+1}) \right) \right\} \tag{3}
\]

Note that I have dropped the dependence of \( v(\cdots) \) on \( \Omega_t \) for ease of exposition.

Let us now characterize the equilibrium strategies in this incomplete information game. Following Milgrom and Weber (1985), define a distributional strategy as a probability measure \( \mu_i \) on the subsets of \( S_i \times Y_i \), for which the marginal distribution of \( S_i \) is \( F_i \).

A pure strategy is defined to be a measurable function \( \overrightarrow{y}_i(s_i) : S_i \rightarrow Y_i \).

For the setup above, Proposition 1 of Milgrom and Weber (1985) guarantees the existence of an equilibrium in distributional strategies. Furthermore, by Proposition 4 of Milgrom and Weber (1985), for each distributional strategy, a pure strategy, \( \overrightarrow{y}_i(s_i) \), exists, which is equivalent to the distributional strategy from a payoff point of view. Another result is obtained by Reny (1999), who shows existence in pure strategies for a discriminatory auction in which quantities are discrete, but prices are continuous.

### 3.1 Necessary conditions: A heuristic derivation

To characterize the necessary condition of optimality for a pure strategy equilibrium, \( \overrightarrow{y}_i(s_i) \), let us look at the effect of deviation \( \Delta y \) of the \( k \)-th component of the bid vector. The benefit of such a deviation will be given by the area labeled “Benefit” in Figure 3, which equals, in expectation

\[
E[\text{Benefit}] = \left( \int_{y_{ik}}^{y_{ik+\Delta y}} v_i(q, s_i) dq - p_k \Delta y \right) [H(p_k, \overrightarrow{y}_i + \Delta y) - H(p_{k-1}, \overrightarrow{y}_i + \Delta y)] \tag{4}
\]

\(^{20}\) \( H(p_k, \overrightarrow{y}_i) \) can also be interpreted as the probability that bidder \( i \) actually wins the \( y_{ik} \) units he requested at price \( p_k \).

\(^{21}\) A distributional strategy can be thought of as an extension of mixed strategies to the incomplete information game.
since the benefit comes from the extra $\Delta y$ quantity that is won *in the event* that the market clearing price is between $p_k$ and $p_{k-1}$, which happens with probability $H(p_k, \bar{y}_i^t + \Delta y) - H(p_{k-1}, \bar{y}_i^t + \Delta y)$.

However, if the market clearing price turns out to be less than $p_{k-1}$, there is no benefit to the bidder, and he ends up paying area labeled “Cost” in Figure 3 with probability $H(p_{k-1}, \bar{y}_i^t + \Delta y)$:

$$E[\text{Cost}] = \Delta y (p_k - p_{k-1}) H(p_{k-1}, \bar{y}_i^t + \Delta y)$$  \hspace{1cm} (5)

For $\Delta y$ not to be a profitable deviation, we should have

$$\frac{1}{\Delta y} \int_{y_{ik}}^{y_{ik} + \Delta y} v_i(q, s_i) dq \leq p_k + \frac{H(p_k, \bar{y}_i^t + \Delta y) (p_k - p_{k-1})}{H(p_k, \bar{y}_i^t + \Delta y) - H(p_{k-1}, \bar{y}_i^t + \Delta y)}$$  \hspace{1cm} (6)

The bound we obtain for the case of a *decrease* in quantity by $\Delta y$ is:

$$\frac{1}{\Delta y} \int_{y_{ik} - \Delta y}^{y_{ik}} v_i(q, s_i) dq \geq p_k + \frac{H(p_k, \bar{y}_i^t - \Delta y) (p_k - p_{k-1})}{H(p_k, \bar{y}_i^t - \Delta y) - H(p_{k-1}, \bar{y}_i^t - \Delta y)}$$  \hspace{1cm} (7)

Recall that in reality, $\Delta y$ corresponds to a request with face value of $6000$ (50 million TL). This corresponds to about 0.00003 of the total supply for the average auction. Therefore I will look at the limit case of the bounds obtained above where $\Delta y \to 0$:

$$v_i(y_{ik}, s_i) = p_k + \frac{H(p_k, \bar{y}_i^t) (p_k - p_{k-1})}{H(p_k, \bar{y}_i^t) - H(p_{k-1}, \bar{y}_i^t)}$$  \hspace{1cm} (8)

Observe that first order condition (8) can be understood more intuitively in terms of the “inverse elasticity markup rule” in monopoly theory. Recall that the monopoly markup is $-\frac{D'(p)}{D(p)}$, the inverse elasticity of demand at the monopoly price. In the divisible good auction setting, each bidder is an *oligopsonist* facing an uncertain residual supply curve, who has some market power to affect the market clearing price with his bid vector.
So if a bidder is asking for \(y_{ik}\) units of T-bills at price \(p_k\), the expected number of T-bills he will receive will be \(H(p_k, y_i)\). If the bidder increases his bid by \(\Delta p\), the change in “expected residual supply” will be \(\frac{\Delta(H(p_k, y_i) \cdot y_{ik})}{\Delta p}\). Using the markup relation, we get

\[
v_i(y_{ik}) = p_k + \frac{H(p_k, y_i)}{H(p_k, y_i) - H(p_k-1, y_i)}(p_k - p_k-1).
\]

### 3.2 Calculation of marginal valuations

Let us take closer look at the first order condition (8) derived in the previous subsection:

\[
v_i(y_{ik}, s_i) = p_k + \frac{H(p_k, y_i)}{H(p_k, y_i) - H(p_k-1, y_i)}(p_k - p_k-1).
\]

If we had a way to estimate \(H(p_k, y_i)\) and \(H(p_k-1, y_i)\), we could calculate bidder \(i\)'s marginal valuation \(v(y_{ik}, s_i)\), that rationalizes his bid of \(p_k\) for \(y_{ik}\) units of the security. In the next section, I will outline a procedure to estimate \(H(p_k, y_i)\) and \(H(p_k-1, y_i)\).

It is worthwhile to discuss briefly what is being calculated. In recent empirical auction literature, considerable attention has been given to precise conditions under which the distribution of unobserved bidder values can be identified (see Guerre et al. (2000), Atthey and Haile (2000)). In the single-unit auction case, the concept of a “private signal” and a “private value” are used in an interchangeable manner; since it is impossible, in general, to identify the mapping from the signal to the value without making additional functional form assumptions. Similarly, observe that the first-order condition above does not say much about the signals: what I calculate are the marginal valuations \(v_i(y_{ik}, s_i)\), which rationalize certain quantity bids observed on the price grid. This might be seen as a weakness in terms of recovering the true economic fundamentals underlying the problem, as the entire mapping of \(s_i\) to \(v_i(q, s_i)\) is not identified and that the marginal valuation function cannot be inverted to identify \(s_i\). However, we should remember that the goal of this paper is to evaluate the revenue performance of an auction mechanism, which requires that we identify the distribution of the marginal valuations rather than the distribution of the signals. Hence, identification of the distribution of bidders’ signals is not relevant for the specific policy application of the paper.\(^{22}\)

### 3.3 The “Missing” Bids

A complication in using the first-order condition (9) in an application using real auction data is that, few, if any, of the bidders submit unique quantity bids for every conceivable point on the price grid. That is, we observe identical quantity requests for consecutive price points \((y_{ik} = y_{ik+1})\).

\(^{22}\)One could as well treat the marginal valuations as the true economic fundamentals of the model rather than resorting to a value specification using signals. This would require the extension of the information structure of the game to allow for multi- or infinite-dimensional signals. Equilibrium existence results such as Milgrom and Weber (1985) apply in general vector spaces. However, to preserve the clarity of exposition, I have chosen to restrict bidder information to be indexed by a scalar signal.
Recall that the discreteness in the price grid is due to rounding of price quotes. Given that typical quoted prices are around 85 TL, and the price grid increment is 0.001 TL, one might think that the absence of different quantity bids for each grid point is just oversight on the part of the bidders.\footnote{In fact, what I find after converting the price offers to interest rate is that quite a few bids are clustered on round percentage points.}

One way to rationalize these “missing” bids is to think of them as the solutions of the optimization problem of the bidder. That is, when choosing the quantity to buy at each price point, the bidder chooses to set his quantity bid equal across some price points. If the bidder’s choices are unconstrained solutions of the optimization problem, we can use equation (9) to calculate the marginal valuations directly.

Another way to rationalize these “missing” bids within an optimization framework is to consider the effect of the monotonicity constraint \( y_{ik} \geq y_{ik+1} \), on the bid vectors explicitly.\footnote{The effect of monotonicity constraints also plays a large part in the analysis of Engelbrecht-Wiggans and Kahn (1998), who analyze a discriminatory auction for two units of a good. They find that with positive probability, the valuations of the bidders will be such that the monotonicity constraint will bind.} The monotonicity constraint introduces a shadow cost of submitting a quantity bid at each price point. At the price points where we do not observe a unique quantity bid, the monotonicity constraint binds, both from above and below. For price points where we do observe a new quantity bid, the monotonicity constraint only binds from below.

More specifically, let \( \{y_{ik1} > y_{ik2} > \ldots > y_{ikL}\} \) be the set of \( L \) “observed” quantity bids that the bidder submits on the price points \( \{p_{k1} < p_{k2} < \ldots < p_{kL}\} \). Once again, the issue is that \( \{p_{k1} < p_{k2} < \ldots < p_{kL}\} \) is a subset of the entire price grid, \( \{p_0 < \ldots < p_K+1\} \). We would like to find the marginal valuation schedule \( v_i(y, s_i) \) that will rationalize the bids we observe. Note that, to make the following characterization, I have to assume that quantities at different price points are continuous choice variables.

Proposition 1 summarizes the “rationalization” condition that explicitly accounts for binding monotonicity constraints:

**Proposition 1.** The following marginal valuation function \( v_i(y, s_i) \) rationalizes the observed quantity bids, \( \{y_{ik1} > y_{ik2} > \ldots > y_{ikL}\} \), assuming that at the price points where we do not observe a unique quantity request, the monotonicity constraint \( y_{ik} \geq y_{ik+1} \) is binding for the bidder:

\[
\begin{align*}
v(y, s_i) &= 0 & y < y_{ik1} - \varepsilon \\
v(y_{ik1}, s_i) &= y_{ik1} - \varepsilon \leq y < y_{ikM-1} - \varepsilon \\
v(y_{ikM-1}, s_i) &= y_{ikM-1} - \varepsilon \leq y < y_{ikM-2} - \varepsilon \\
\vdots \\
v(y_{ik2}, s_i) &= y_{ik2} - \varepsilon \leq y < y_{ik1} - \varepsilon \\
v(y_{ik1}, s_i) &= y_{ik1} - \varepsilon \leq y \leq y_{ik1} + \varepsilon \\
v(y_{ik1}, s_i) &= 0 & y > y_{ik1} + \varepsilon
\end{align*}
\]
where, \( v_i(y_{ik^1}, s_i), ..., v_i(y_{ik^L}, s_i) \) is the solution of a (recursive) set of linear equations. Assuming that \( \varepsilon \) can be made arbitrarily small, these equations are, for \( 1 < m < L \):

\[
v_i(y_{ik^m}, s_i) = p_{km} + \frac{H(p_{km-1}, \bar{y}_i') [p_{km} - p_{km-1}]}{H(p_{km}, \bar{y}_i') - H(p_{km-1}, \bar{y}_i')} + \frac{B_{km} - A_{km}}{H(p_{km}, \bar{y}_i') - H(p_{km-1}, \bar{y}_i')}
\]

where

\[
A_{km} = \frac{\partial H(p_{km}, \bar{y}_i')}{\partial y_{ik^m}} \{ v_i(y_{ik^m+1}, s_i) - p_{km} \} (y_{ik^m} - y_{ik^m+1})
\]

\[
B_{km} = \frac{\partial H(p_{km-1}, \bar{y}_i')}{\partial y_{ik^m}} \{ v_i(y_{ik^m}, s_i) - p_{km} \} (y_{ik^m+1} - y_{ik^m})
\]

and for \( m = 1 \):

\[
v_i(y_{ik^1}, s_i) = p_{k1} - \frac{\partial H(p_{k1}, \bar{y}_i')}{\partial y_{ik^1}} \{ v_i(y_{ik^2}, s_i) - p_{k1} \} (y_{ik^1} - y_{ik^2}) / H(p_{k1}, \bar{y}_i')
\]

and for \( m = L \):

\[
v_i(y_{ik^L}, s_i) = p_{kL} + \frac{H(p_{kL-1}, \bar{y}_i') [p_{kL} - p_{kL-1}]}{H(p_{kL}, \bar{y}_i') - H(p_{kL-1}, \bar{y}_i')} + \frac{B_{kL} - A_{kL}}{H(p_{kL}, \bar{y}_i') - H(p_{kL-1}, \bar{y}_i')}
\]

\[
A_{kL} = -\frac{\partial H(p_{kL}, \bar{y}_i')}{\partial y_{ik^L}} p_{kL} y_{ik^L}
\]

\[
B_{kL} = \frac{\partial H(p_{kL-1}, \bar{y}_i')}{\partial y_{ik^L}} \{ v_i(y_{ik^L}, s_i) - p_{kL-1} \} (y_{ik^L-1} - y_{ik^L})
\]

Proof. See Appendix A.1.

Observe that the rather complicated expressions in Proposition 1 indicate that the monotonicity constraint leads to “coupling” between consecutive quantity bids on the price grid. That is, a bidder’s decision whether to submit a different quantity bid at a given price point also depends on his decision for the next price point. This is the main result of the analysis in Engelbrecht-Wiggans and Kahn (1998), who argue that the monotonicity constraint causes “pooling” of the bids in a discriminatory auction.

Fortunately, a careful look at the system of equations in Proposition 1 reveals that given estimates of \( H(p_{km}, \bar{y}_i') \) and \( \partial H \) \( / \partial y_{ik^m} \) for all \( k^1, ..., k^L \), the marginal valuations \( v_i(y_{ik^1}, s_i), \cdots, v_i(y_{ik^L}, s_i) \) can be calculated directly.

Figure 4 illustrates this “rationalization” condition. In this illustration, there are 10 price points that the bidder can place a quantity bid on, \( p_0, \cdots, p_9 \). However, instead of submitting a quantity bid for each price point, the bidder has given 3 price-quantity pairs, \( \{(p_1, y_1), (p_4, y_4), (p_7, y_7)\} \). In the figure, I mark these price-quantity pairs as the solid circles, and use the notation in Proposition 1, i.e. \( p_1 = p_{k^1}, y_1 = y_{ik^1} \), etc.
In Figure 4, I also mark the marginal valuations that rationalize the observed bids, given the conditions in Proposition 1. Observe that this is a step function, which is 0 for quantities less than $y_{ik3}$ and greater than $y_{ik1}$. The marginal valuations $v_i(y_{ik1}), v_i(y_{ik2}), v_i(y_{ik3})$ that comprise the steps of this function can be calculated using the system of equations in Proposition 1.

A possible issue illustrated by Figure 4 and the argument in Proposition 1 is that the marginal valuation function that rationalizes the bids is not necessarily uniquely determined by the finitely many points we observe in the data. This makes a lot of sense, as “identifying” a function uniquely from only a finite set of points is not a plausible objective.

Another issue is that the “step function” that rationalizes the observed bids is defined up to $\varepsilon$, which is essentially a technical device to eliminate discontinuities in the marginal valuation function at the quantity points that we observe. Unfortunately, $\varepsilon$ is not determined by any observables. Hence, in practice, we can only calculate a $\varepsilon$ approximation to the marginal valuation function. However, in any empirical application, the standard errors associated with estimation will swamp the indeterminacy introduced by $\varepsilon$.

25. Another way to think of this is to recall that bidders are in fact constrained to discrete quantities. Hence, an indeterminacy due to these discrete quantity bids becomes obvious. I should also note that in any empirical auctions application, the empirical researcher has to deal with the fact that the game is not entirely continuous and that bids have to be rounded to dollars or cents.
4 The Empirical Strategy

In the last section we saw that the “interior” first-order condition of equation (9):

\[ v_i(y_{ik}, s_i) = p_k + \frac{H(p_k, \bar{y}_i^k) (p_k - p_{k-1})}{H(p_k, \bar{y}_i^k) - H(p_{k-1}, \bar{y}_i^k)} \]  

(10)
or the more complicated system of equations of Proposition 1, which uses information regarding the monotonicity constraint, can be utilized to calculate the marginal valuations of the bidders at certain quantity points. To operationalize this theoretical result to estimate \( v_i(y_{ik}, s_i) \), we have to find a way to reconstruct a bidder’s equilibrium belief about the probability distribution of the market clearing price conditional on his bid vector, \( H(p_k, \bar{y}_i^k) \).

One way to approach this problem is to ask how a bidder would assess \( H(p_k, \bar{y}_i^k) \). In an equilibrium of the game, bidder \( i \) knows other bidders’ bid vectors up to their signals, \( s_j, j \neq i \). Hence, in equilibrium, the residual supply curve that bidder \( i \) faces is a function of \( N - 1 \) random variables, each of whose distribution is common knowledge among bidders. But bidder \( i \) could simulate the gamut of residual supply curves he is likely to encounter by repeating the following procedure many times: First, generate a random draw of the \( N - 1 \) element signal vector of the competing bidders, \( j \neq i \). Then, evaluate the \( N - 1 \) equilibrium “opponent” bids \( y_k(s_j) \) corresponding to these signals. Using these, form the residual supply function, \( RS_k = Q - \sum_{j \neq i} y_k(s_j) \). Calculate the market clearing price by finding the intersection of the residual supply function with own bid vector.

After several thousand iterations, the above procedure will get the bidder a very good approximation to \( H(p_k, \bar{y}_i^k) \). Furthermore, in equilibrium, it must be the case that the bid vector \( \bar{y}_i \), is an optimal response to the distribution of residual supply functions bidder \( i \) believes he is going to face – hence \( \bar{y}_i \) should satisfy the first order condition.

Unfortunately, the distribution of bidders’ signals is not observed by the empirical researcher. More importantly, the exact mapping from the signals, \( s_i \) to the equilibrium bid vectors \( \bar{y}_i \) is not available in closed form except in a very special case analyzed in Appendix C. A computational procedure utilizing the “fixed-point correspondence” nature of the first order condition in equation (9) could be formulated; however, I have not been able to obtain a contraction mapping result that guarantees a unique solution to the problem.26 Given the practical constraints, I will assume that the equilibrium mapping from the signals to the bid vectors can not be inverted explicitly by the econometrician to pursue a parametric, likelihood based approach.27

Fortunately, the following assumption goes a great distance in coming up with an empirical procedure to estimate \( H(p_k, \bar{y}_i^k) \):

26 However, I am continuing to investigate computational approaches to solving the equilibrium of the game explicitly.

27 Some examples of parametric, likelihood-based approaches to estimate structural models of single-unit auctions are Paarsch (1992), Laffont, Ossard and Vuong (1995), Bajari and Hortacsu (2000).
Assumption 2. Bids in the data are generated by a symmetric pure strategy equilibrium of the independent private value discriminatory price auction. That is, $\bar{y}_i^k(s) = \bar{y}(s)$.

Note that although existence was obtained for pure strategy equilibria $\bar{y}_i^k(s)$, existence of symmetric pure strategy equilibria, $\bar{y}(s)$, is not guaranteed, even in the case when the marginal valuation functions are identical across bidders. However, this assumption might not be a very strong one, as Corollary 5.3 of Reny (1999) establishes the existence of a symmetric pure strategy equilibrium of an independent private value discriminatory auction game in which the price axis is continuous, but the quantities are discrete.\(^{28}\)

Assumption 2 yields the following:

**Lemma 1.** $H(p_k, \bar{y}_i^k) = \Pr\{p_k^* \leq p_k \mid \bar{y}_i^k\} = \Pr\{\frac{1}{N-1} \sum_{j \neq i}^N y_k(s_j) \leq \frac{Q - y_k(s_i)}{N-1} \mid s_i\}$. That is, the probability distribution of the market clearing price can be represented as the probability distribution of the sample mean of $N - 1$ i.i.d. random variables, $\{y_k(s_1), ..., y_k(s_{i-1}), y_k(s_{i+1}), ..., y_k(s_N)\}$.

**Proof.** See the Appendix. \(\square\)

Given this result, I propose an estimator for $H(p_k, \bar{y}_i^k)$ using the intuition underlying various resampling methods used in statistics: when one wants to estimate the sampling distribution of a statistic about its true value, resampling with replacement from the data and recomputing the statistic for these resamples will approximate the true sampling distribution of the statistic.

The basic resampling procedure I propose to undertake is the following:

1. Fix bidder $i$ among the total $N_t$ bidders in auction $t$.
2. From the sample of $L = \sum_{t=1}^T N_t$ bid vectors in my data set, draw a random sample of $N_t - 1$ bid vectors with replacement, giving equal probability of $\frac{1}{L}$ to each bid vector in the original sample.
3. Construct the residual supply function generated by these $N_t - 1$ “resampled” bid vectors.
4. Intersect with bidder $i$’s bid to find the market clearing price.
5. Repeat $B$ times for each bidder and for all bidders in the data set.

This generates $B$ market clearing prices conditional on $\bar{y}_i^k$. I then estimate $H(p_k, \bar{y}_i^k)$ by counting the frequency with which a given $p_k$ remained above the market clearing prices generated above.

Let us call the above estimator of $H(p_k, \bar{y}_i^k)$, $\hat{H}^R(p_k, \bar{y}_i^k)$, the resampling estimator of the probability distribution of the market clearing price. The following result can be proved using arguments from probability theory:

---

\(^{28}\)I am currently working on exploring the implications of Reny’s setup on the characterization of the first-order conditions of my problem.
**Proposition 2.** Suppose we observe $L$ bid vectors $\{\vec{y}(s_1), \cdots, \vec{y}(s_L)\}$ generated by $L$ i.i.d. draws from the distribution of private bidder signals, where $y(s)$ denotes the symmetric pure strategy equilibrium of the discriminatory price auction with $N$ bidders. Then, as $L \to \infty$, $\hat{H}^R(p_k, \vec{y}_i)$ converges to $H(p_k, \vec{y}_i)$ almost surely.

**Proof.** The basic intuition for the consistency of the resampling procedure is to see that resampling is a way to make random draws from the empirical distribution of data. By the Glivenko-Cantelli Theorem, the empirical distribution converges (almost surely) to the true distribution as the sample size grows large. See the Appendix for a more detailed argument.

After $\hat{H}^R(p_k, \vec{y}_i)$ is obtained, we can plug the estimate into equation (9) or the system of equations in Proposition 1 to calculate $\hat{v}(y_{ik}, s_i)$. To use the equations in Proposition 1, we also need to estimate the partial derivative of $H$ with respect to $y_{ik}$. This can be done by a finite difference approximation, in which we set a $\delta y$ and estimate $H(p_k, y_{ik} + \delta y)$ and $H(p_k, y_{ik} - \delta y)$, and then approximate the derivative by $\frac{H(p_k, y_{ik} + \delta y) - H(p_k, y_{ik} - \delta y)}{2\delta y}$.

Observe that these marginal value estimates do not depend on functional form or distributional assumptions. The resampling procedure to simulate the conditional distribution of the market clearing price is fully data driven. Another advantage of this procedure is that I do not need to compute the equilibrium bid function to back out the valuations. Hence, this is an “indirect” procedure, similar to Elyakime et al. (1994) and Guerre et al. (2000). A “direct” approach to this problem, as suggested, for example, by Armantier, Florens and Richard (1997) in the context of single unit auctions would be to fix a parametrization of the signal distribution, compute the equilibrium bid functions corresponding to this signal distribution, and minimize over the parameter space a metric between the computed bids and the observed bids. Such a method is computationally much more complex in the absence of analytically characterizable equilibrium bid functions and is dependent on particular functional form assumptions regarding the distribution of private signals and demand functions of the bidders.

The correct asymptotic condition in Proposition 2 is not the one in which the number of bidders is approaching infinity, but in which the number of bid vectors we observe for an $N$-person auction (the number of data points) is going to infinity.

Observe that in the statement of the resampling procedure and in the asymptotic argument, I have implicitly assumed that auctions take place in a static environment in which the state vector $\Omega_t$ stays constant across auctions, but bidders draw different signals in each new auctions. If there are $T$ such auctions, then the number of bid vectors we can resample from will be $L = TN$.

The assumption of an essentially static environment ($\Omega_t$ constant) can be realistic in certain divisible good auctions settings like electricity markets, in which the auction is run every day, sometimes multiple times within a day. However, in the Treasury auction setting, it is reasonable to expect that the economic environment
surrounding the auction changes in time. For example, the summary statistics in Table 2 show that the number of bidders in the auction and the supply of securities vary greatly across auctions in my data set.

One way to account for a non-constant auction environment is to condition on \( \Omega_t \) when constructing the empirical distribution of bid vectors. If the dimension of \( \Omega_t \) is small, kernel methods and other non-parametric smoothing techniques can be used for this purpose, as in Elyakime et al. (1994) and Guerre et al. (2000). The main drawback of this method is that the dimension of conditioning variables in \( \Omega_t \) has to be small to avoid the “curse of dimensionality problem.” I have also found that the selection of bandwidth parameters for the smoothing kernels is also a delicate matter in this problem, due to the bias introduced by these parameters. A computational difficulty with this approach is that one has to make simulations draws from a smoothed estimate of the conditional distribution of bid vectors. This slows down the resampling scheme considerably. To address this, I am exploring semi-parametric specifications to estimate the distribution of \( y_{ik} \)'s – the observed (or implied) quantity bids for each price point, \( p_k \). An exponential specification for this distribution seems to fit the data well, and is also a relatively easy parametric specification to make simulation draws from.

A simpler solution is to restrict attention to data from a single auction, without “pooling” data across auctions. This does away with having to condition out \( \Omega_t \). However, it is disadvantaged from the perspective of the number of data points it utilizes, since \( T = 1 \) and \( L = N \) in this case. In spite of this disadvantage, however, the combinatorics underlying the resampling procedure begins to play an important role if \( N \) is large enough. In my data set, there are, on average, 70 bidders in each auction. Hence, the number of ways in which the residual supply curve for a given bidder can be constructed using resamplings of the bid vectors with replacement is approximately \( 1.18 \times 10^{40} \). Therefore, even with data from a single auction it is possible to generate many unique simulation draws of the market clearing price.

Does \( \hat{H}^R(p_k, \overline{y}_{i}) \) provide a good approximation for \( H(p_k, \overline{y}_{i}) \) with \( L = N = 70 \)? Here is where a “bootstrap” analogy to my resampling scheme plays an important role in building intuition for the small sample performance of the resampling procedure: When bootstrapping the distribution of a sample mean, for example, the statistician has only one direct observation of the sample mean of \( N \) data points, i.e. \( T = 1 \) and \( L = N \), just as in this problem. However, by resampling from the \( N \) data points at hand, and computing the mean of these resamples, the statistician can build up an approximation of the distribution of the sample mean, centered around the one direct observation of the sample mean. It has been shown in the statistics literature that for \( N \) sufficiently large, a resampling based “bootstrap approximation” for the distribution of a sample statistic is often as good or better than the classical normal approximation.

In the auctions that I analyze, \( N = 70 \) on average. To see if this is enough for my application, I conduct

\[ \text{The formula for general } N \text{ is } \frac{2N - 1}{N}. \text{ See Hall (1992), Appendix I. The formula is not } N^N \text{ since ordering of bid vectors within a resample does not matter.} \]

\[ \text{See Efron and Tibshirani (1998) and Hall (1992) for detailed accounts.} \]
a simulation experiment, in which I use data from a simulated auction for which I know (or can compute) the true marginal valuations, and see how estimated marginal valuations using data from only a single auction matches up with the known true marginal valuations of the bidders. The next section discusses the details of this experiment. In Appendix B, I also discuss two ways to assess the standard errors of \( \hat{H}^R(p_k, \overrightarrow{y}_i) \).

### 4.1 A simulation study

To validate the small sample performance of the resampling scheme proposed above, I conduct a simulation study in which I compare estimated marginal valuations with the known “true” marginal valuations of the bidders.

Unfortunately, as mentioned in the previous section, explicit computation of equilibrium strategies for the discriminatory price auction game is analytically not possible, except for the simple case analyzed in Appendix C. Therefore, I took the approach of assuming a particular functional form for the equilibrium strategies, and constructed the “true” valuations that would rationalize these strategies by the simulation procedure outlined for the bidder in section 4.

Inspection of the data reveals that bid vectors in the price quantity plane can be interpolated quite accurately with linear functions. Therefore, for my experiment, I took a market with \( N = 70 \) bidders who I assumed to be following symmetric linear bid strategies:

\[
y(p_k, s_i) = \alpha + \beta s_i - \gamma p_k
\]  

(11)

I allowed the bidders to submit quantity bids on 20 price points regularly spaced between a minimum price of 0.815 and a maximum of 0.8654.\(^{31}\) I set \( \alpha = 0.71, \beta = 1, \gamma = -0.83 \) and \( Q = 1 \), and I specified the private signals to be i.i.d. normally distributed with zero mean and standard deviation 0.015. With the given constants, the resulting bid functions are roughly calibrated to resemble linear approximations to the bid functions in the data set.

For demonstration purposes, I take one realization of this auction – for which I took \( N = 70 \) draws from the distribution of \( s_i \). This gave me 70 bid functions, which I treat as my “data.”

To calculate the “true” marginal valuations corresponding to the “data,” I followed the “bidder’s procedure” outlined in the previous section: I simulated 10,000 residual supply functions for each bidder by making draws from the known distribution of the signals. I then plugged this distribution into the first-order condition in Proposition 1 to calculate the true marginal valuation corresponding to each bid.

Once again, in the empirical exercise, we do not know the distribution of \( s_i \)’s. All we see is data from one auction – in this case, the 70 bid functions in the generated data set. The object is to see if estimating the

\(^{31}\)In the data set, the range of prices is between 0.83 and 0.88.
marginal valuations by bootstrapping these 70 bid functions will be successful in recovering the “true” marginal valuations.

I proceeded exactly in the “empirical procedure” proposed in the previous section: Holding one bidder’s bids fixed at a time, I draw 10000 samples of size 69 from the empirical distribution of bid vectors. I then calculate the residual supply curve that these resampled bids generate, and calculate the probability distribution of the market clearing price that these residual supply curves and the fixed bidder’s bid vector yield. I then plug point estimates of this probability distribution into the “interior” first-order condition in 9 to calculate the marginal valuation of the bidder.\textsuperscript{32}

Figure 5 displays the result of this experiment. The thick line plotted with x’s is the aggregate bid schedule in this auction, the solid line is the “true” marginal valuation calculated using draws from the known signal distribution, and the broken line is the estimated marginal valuation calculated using the bootstrap method. We see that the estimated marginal valuations are quite close to the true marginal valuations for a wide range of prices about the actual market clearing prices.

Repeating this experiment 200 times with different “data sets” did not yield very different results from this demonstration. Hence, I conclude that the performance of the estimation method is likely to be quite good.

\textsuperscript{32}The “interior” first order condition is valid in this case since bidders submit different quantities on every price point.
To demonstrate the results that are obtained from each step of the estimation algorithm developed in section 4, I will start by showing the results for the first auction in my dataset, the 3-month T-bill auction of October 1991. In this auction, there were 67 bidders for what amounted to 1959.5 billion Turkish liras worth of Treasury bills – about 400 million U.S. dollars. 58% of the bids that were submitted were successful. The market clearing price was 84.388 TL for a 100TL face value 3 month T-bill.

I will focus on bidder #2, who submitted 14 price-quantity pairs totalling to a demand of 12% of the issue. In figure 6, I superimpose the histogram of market clearing prices, which resulted from 20000 resampled draws of the residual supply curve, horizontally on top of bidder #2’s “staircase” bid vector. The vertical axis denotes the price in TL. The horizontal axis gives the quantity that bidder #2 requested as a percentage of the total supply. It also gives the frequency distribution of the simulated market clearing prices for the histogram.

We see that except for the highest bid at 84.825 TL, all other bids lie within the support of market clearing price distribution. This distribution is not very smooth, as would be expected from the “staircase” nature of the bids.

Just looking at this figure, we see that the high bid at 84.825 TL is an anomaly with potential economic significance. First of all, this bid looks like an outlier among the next 13 bids, which are much closer together. It could be that this is a mistaken bid, for if we take the estimated distribution of market clearing prices at face
value, then it is not rational for bidder #2 to bid this high a price.

Another explanation for this high bid is that the bidder has a really high value for the initial “step” of quantity. Such a high valuation makes sense when we consider that banks in Turkey have to satisfy a liquid asset reserve requirement which is monitored very closely by the Central Bank. If a bank can not win enough T-bills in the auction, then it has to buy securities in the resale market or in the following week’s auction to close its reserve shortfall. Since the resale market had much less volume than the primary market in this period, the bank could run the risk of falling short of its reserve requirement, and paying a punishment fee to the government or face suspension of its operations. In fact, market participants have reported that in some banks, two different departments are in charge of preparing the bid vector: the department in charge of treasury operations puts in several bids to meet the reserve requirement, along with more “speculative” bids given by the trading desk.

To calculate the marginal valuations of bidder #2 that correspond to each price-quantity pair submitted, I use the set of equations in Proposition 1. However, in practice, I found that the partial derivatives, $\frac{\partial H(p_{km}, \vec{y})}{\partial y_{ikm}}$, were identically zero in almost all instances. This eliminates the “coupling” introduced by the terms $A$ and $B$ in the expressions in Proposition 1.\(^{33}\) This makes sense since for small perturbations in $y_{ik}$, the distribution of the market clearing price is not affected very much in this auction with $N = 67$ bidders.\(^{34}\) Given this finding, I use the “decoupled” set of equations to find the marginal valuations rationalizing the 14 bids of bidder #2:

$$v_i(y_{ikm}, s_i) = p_{km} + \frac{H(p_{km-1}, \vec{y}_{i})[p_{km} - p_{km-1}]}{H(p_{km}, \vec{y}_{i}) - H(p_{km-1}, \vec{y}_{i})}$$

for $1 < m \leq 14$, and

$$v_i(y_{ik1}, s_i) = p_{k1}$$

for $m = 1$. I use a simple counting estimator for $H(p_k, \vec{y}_i)$ and computed standard errors using both the “jackknife-after-bootstrap” and “analytic” methods, which are discussed in Appendix B. The standard error estimates are comparable in most cases I have investigated. The reported standard errors are calculated using the “analytic” method.

Figure 7 displays the estimation results for bidder #2, using 20000 realizations of the simulated market clearing price. The horizontal axis is once again the percent of total supply that bidder #2 requested, with the prices on the vertical axis. Once again, we have the “staircase” representation of the bid vector. The points marked with a “v” correspond to the point estimates of the marginal valuation for the quantity bids, calculated using the equations above. I also plot the 5-95% confidence band around my point estimates.

\(^{33}\)Since $H$ is estimated with error, I repeated the estimation procedure 50 times and recalculated the partial derivatives using the finite-difference method.\(^{34}\) However, this might not be true in situations where there only a few bidders, or “$\delta y$” is large compared to the total quantity which is the case in Engelbrecht-Wiggans and Kahn (1998), where $\delta y = 1$ and total quantity is 2.
Figure 7: Estimation results for bidder #2 in auction #1

Excluding the marginal value estimate for the first and last price-quantity pairs, marginal valuations are on average 0.16 TL (about 0.15% in annual yield terms) higher. The confidence intervals are wide for two intermediate quantities, where the point estimates of the marginal valuation seem to go against declining marginal value – given these wide confidence bands, the estimated marginal valuations can be safely seen as being declining in quantity. The confidence band is very narrow for the rightmost two bids, and the rightmost valuation estimate is very close to the bid price. From Figure 7 we see that the probability that the market clearing price falls below the lowest two bid prices is very low, hence it makes sense that the bidder should have a valuation close to his bid if he is almost certain that he will never win the auction.

As for the first (leftmost) bid, we see that the marginal valuation that rationalizes this is indeed quite high, in concordance with the intuition that banks with reserve price requirements may indeed value initial units very highly.

5.1 Counterfactual comparisons

To perform a counterfactual revenue comparison between the uniform price auction and the discriminatory price auction, I first repeat my analysis for bidder #2 for every bidder in auction #1. This gives me a set of point estimates for the marginal valuations rationalizing each price quantity pair observed in this auction.

One way to do the counterfactual comparison is to use the point estimates of the marginal valuations as
the true marginal valuations of the bidders. This yields the perfectly competitive outcome of this auction, in which bidders reveal their true marginal valuations. The revenue of the auctioneer in the competitive outcome provides an upper bound to the revenue from the uniform price auction, since bidders typically exercise “demand reduction” in the uniform price auction. The competitive outcome also provides an upper bound to the revenue from the Vickrey auction.

Since I estimate only certain points on the marginal value function, I have to account for the marginal valuations in between the points I estimate. Proposition 1 resolves this by restricting the marginal valuation to be the “upper envelope” of the pointwise estimates of the marginal valuations. In line with this proposition, I obtain the “upper envelope” of the step function interpolation of the marginal value estimates. As a robustness check, I also calculate the “lower envelope” of the point estimates of the marginal valuations. For the point estimates in Figure 7, the upper and lower envelopes are illustrated in Figure 8. With a general weakly decreasing marginal valuation function connecting these point estimates, the upper (lower) envelope provides an upper (lower) bound as to what the marginal value function should be. Comparing Figure 7 to Figure 8, we can see that except for the first bid, the envelope of the marginal valuation lies within the confidence intervals of the estimates.

Using the upper envelope for each bidder’s marginal valuation functions, I can calculate the revenue in the best-case uniform price auction by finding the market clearing price that would have resulted if the bidders
had revealed their marginal valuations truthfully. To do this, I aggregate the upper envelope of the estimated marginal valuations across bidders, and find the intersection of this aggregate schedule with the total supply.

Figure 9 illustrates the result of this procedure for auction #1. In this figure, I overlay the aggregated bid schedule and the aggregated upper envelope of the marginal valuations. With supply normalized to 1 in this figure, the counterfactual uniform price auction revenue is given by the rectangle formed by the intersection of the aggregate marginal valuation schedule and total supply. Since the area under the aggregate bid schedule up to the total supply is greater than this rectangle, the actual discriminatory price auction revenue is greater than the counterfactual uniform price auction revenue. Specifically, I calculate that the market clearing price in the counterfactual uniform price auction would be 84.465 TL, as opposed to 84.388 TL in the discriminatory price auction. The revenue in the counterfactual case would be 1.3271 trillion TL, as opposed to the actual revenue of 1.3522 trillion TL generated by the discriminatory price auction, which is about a $5.2 million revenue loss. In percentage terms, the counterfactual uniform price auction causes a revenue loss of 1.86%.

A more careful way to do the counterfactual revenue comparison is to remember that according to our model, the bid vectors seen in the data, and the estimated marginal valuations are random variables conditional on environmental variables contained in \( \Omega_t \). Therefore, auction revenues from both mechanisms are also random variables conditional on \( \Omega_t \). Hence, a better comparison of the two mechanisms would be compare the expected revenues from each mechanism.
To compare the expected revenues under both mechanisms, I once again use a resampling procedure. For the discriminatory price auction, I draw 10000 resamples from the original set of bid vectors and calculate the auctioneer's revenue with the resampled set of bids. I do the same with the “best-case” uniform price auction, where my estimated marginal valuation estimates as the bids.

Table 3 reports the results of the two counterfactual experiments for each auction in the data set. The first counterfactual experiment is the ex-post revenue comparison that treats the actual revenue in a given auction and the revenue that would have been generated in a uniform price auction in which the bids were the estimated marginal valuations. The second counterfactual experiment is the ex-ante revenue comparison that compares the expected revenues from both mechanisms.

We see that in all ex-post comparisons auctions, the “best case” uniform price auction would have yielded lower revenues for the Treasury. The average revenue loss would have been 3.8% of the total revenue, or about $ 9 million per auction. According to these estimates, the cost of a switch to the uniform price auction would have been about 450 million dollars out of the 12 billion dollars of debt that was auctioned in this period!

The ex-ante comparison yields a similar result. However, observe that the standard errors on the ex-ante figures are rather high, giving a statistical significance level of about 80% to the reported revenue losses from the switch to the uniform price auction. However, it is once again striking to see that for all auctions without exception, the point estimate of the ex-ante expected revenue from a discriminatory price auction is higher than in the “best-case” uniform price auction.

I should add that these revenue comparisons also say something about counterfactual revenues in Vickrey auctions. In a private values setting, it is well-known that the Vickrey auction achieves truthful revelation of marginal valuations. Therefore one might be curious about the revenue performance of this mechanism. The answer to this question can be gleaned from a comparison of the payment rules in the uniform price and the Vickrey auctions: if bidders reveal their true marginal valuations in the uniform price setting, the revenue from the uniform price auction should be an upper bound to the revenue from a Vickrey auction, since the residual supply curve is increasing. Hence, the results in Table 3 indicate that the discriminatory price auction also outperforms the Vickrey auction in terms of revenue.

5.2 Discussion of the results

What drives the apparent revenue advantage of the discriminatory price auction? In section 5.1, I alluded to the fact that binding liquid asset reserve requirement could cause bidders to value a certain minimum quantity of Treasury bills much higher than subsequent units. For example, in Figure 6, it is clear that bidder #2 values the initial units he wins much higher than the subsequent units – and this hunch is verified in Figure 7.

If the market clearing price in an auction is in general set by bids on marginal/lower-valued units, a

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35See Ausubel and Cramton (1998b) for a discussion of the Vickrey auction.
Table 3: Counterfactual Revenue Comparisons

<table>
<thead>
<tr>
<th>Auction</th>
<th>Date</th>
<th>Revenue (million $)</th>
<th>Ex-post(^a)</th>
<th>Ex-ante</th>
<th>Std. dev. of ex-ante Rev. loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10/16/91</td>
<td>404</td>
<td>0.30%</td>
<td>2.68%</td>
<td>(3.40%)</td>
</tr>
<tr>
<td>2</td>
<td>11/13/91</td>
<td>388</td>
<td>3.10%</td>
<td>27.22%</td>
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</tr>
<tr>
<td>3</td>
<td>12/11/91</td>
<td>415</td>
<td>1.00%</td>
<td>0.12%</td>
<td>(5.07%)</td>
</tr>
<tr>
<td>4</td>
<td>01/08/92</td>
<td>368</td>
<td>2.49%</td>
<td>4.27%</td>
<td>(4.85%)</td>
</tr>
<tr>
<td>5</td>
<td>02/05/92</td>
<td>61</td>
<td>1.17%</td>
<td>5.81%</td>
<td>(6.22%)</td>
</tr>
<tr>
<td>6</td>
<td>03/04/92</td>
<td>45</td>
<td>14.77%</td>
<td>10.22%</td>
<td>(10.32%)</td>
</tr>
<tr>
<td>7</td>
<td>04/01/92</td>
<td>146</td>
<td>2.37%</td>
<td>1.49%</td>
<td>(4.48%)</td>
</tr>
<tr>
<td>8</td>
<td>04/29/92</td>
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<td>1.47%</td>
<td>25.59%</td>
<td>(31.38%)</td>
</tr>
<tr>
<td>9</td>
<td>05/27/92</td>
<td>216</td>
<td>0.24%</td>
<td>1.48%</td>
<td>(2.40%)</td>
</tr>
<tr>
<td>10</td>
<td>06/24/92</td>
<td>322</td>
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<td>(8.60%)</td>
</tr>
<tr>
<td>11</td>
<td>07/22/92</td>
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<td>0.25%</td>
<td>3.99%</td>
<td>(5.19%)</td>
</tr>
<tr>
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<td>0.18%</td>
<td>2.01%</td>
<td>(2.10%)</td>
</tr>
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<td>13</td>
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<td>(2.39%)</td>
</tr>
<tr>
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<td>(4.20%)</td>
</tr>
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<td>(3.58%)</td>
</tr>
<tr>
<td>16</td>
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<td>(6.47%)</td>
</tr>
<tr>
<td>17</td>
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<td>9.27%</td>
<td>(11.36%)</td>
</tr>
<tr>
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<td>5.03%</td>
<td>(6.52%)</td>
</tr>
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<td>19</td>
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<td>179</td>
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<td>14.13%</td>
<td>(16.04%)</td>
</tr>
<tr>
<td>20</td>
<td>04/28/93</td>
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<td>5.19%</td>
<td>(5.92%)</td>
</tr>
<tr>
<td>21</td>
<td>05/26/93</td>
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<td>24.79%</td>
<td>(25.09%)</td>
</tr>
<tr>
<td>22</td>
<td>06/23/93</td>
<td>196</td>
<td>0.52%</td>
<td>14.40%</td>
<td>(14.73%)</td>
</tr>
<tr>
<td>23</td>
<td>07/21/93</td>
<td>394</td>
<td>3.03%</td>
<td>15.29%</td>
<td>(15.21%)</td>
</tr>
<tr>
<td>24</td>
<td>08/18/93</td>
<td>73</td>
<td>22.05%</td>
<td>26.01%</td>
<td>(28.13%)</td>
</tr>
<tr>
<td>25</td>
<td>09/15/93</td>
<td>71</td>
<td>2.71</td>
<td>16.13</td>
<td>(21.15%)</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td><strong>482</strong></td>
<td><strong>3.80%</strong></td>
<td><strong>14.23%</strong></td>
<td></td>
</tr>
</tbody>
</table>

\(^a\)From switching to uniform price auction

\(^b\)The 03/03/93 auction was effectively canceled since only 0.23% of bids appear to have been awarded.
uniform price auction fails to capture the high surplus gained by bidders on inframarginal units. Would a discriminatory price auction do better in such a situation? Bidders who foresee that marginal units will play a greater role in setting the clearing price will, of course, shade their bids for inframarginal units in the discriminatory price auction. However, if there is positive probability that the market clearing price will be high enough to endanger winning the inframarginal units, a bidder cannot justify shading his bid in this region too much. Hence the discriminatory auction can extract more bidder surplus in the inframarginal part of the demand curve than the uniform price auction.

To illustrate this intuition, let us take another look at Figures 6 and 7: Bidder #2 might be tempted to put in a much lower price for the first “step” of his bid schedule, since most of the probability mass for the market clearing price falls around 84.3 TL. However, the fact that the market clearing price still has positive probability of being higher than, say, 84.5 TL, restricts his ability to shade his bid much lower than 84.8 TL. But based on the valuation estimates in Figure 7, we can extrapolate that the market clearing price in a uniform price auction will, on average, still be set by marginal units, which bidder #2 values below 84.5 TL (we found in Section 5.1 that it would actually be 84.465). Hence the uniform price auction would fail to capture any of bidder #2’s surplus from winning the inframarginal units.

To investigate the hypothesis that liquid asset reserve requirements drive the difference between inframarginal and marginal units, I have to utilize information about individual liquid asset reserve requirements. Unfortunately, bidders in my data set are anonymized; therefore I cannot access bidder-specific information from other sources. However, I can exploit the panel structure of my data set, since bidder IDs remain constant through the panel.

Since I observe the quantity demanded by each bidder in a given auction, and since I also see the quantity won by each bidder, I form the variable \( \text{SHORTFALL}_{t-1} \) to denote the percentage of demand shortfall that the \( i \)-th bidder experienced in auction \( t-1 \). This variable acts as a proxy for the severity of the liquid asset reserve requirement of the bidder coming into the current auction, since bidders do not have many other venues to fulfill their reserve needs (as discussed in section 3, the secondary market is not very liquid compared to the auction market in the first half of the sample).

The main problem with this proxy is that between two consecutive 3-month Treasury bill auctions, there are 3 more auctions of 6,9 and 12-month maturities each. Unfortunately, I do not have bidding data for these intermediate auctions. Hence, I can not observe, for example, whether a bidder made up for his security shortfall by bidding in these auctions.

To investigate the effect of the percentage demand shortfall in the previous auction on elasticity of a bidder’s marginal valuation in the current auction, I calculate the average elasticity of a bidder’s marginal valuation by averaging the “step-wise” elasticities implied by the upper-envelope of the pointwise estimates of
Table 4: Variable definitions

<table>
<thead>
<tr>
<th>Dependent Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ELASTICITY</strong></td>
<td>Elasticity of the upper envelope of bidder’s estimated marginal valuation</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Independent Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SHORTFALL</strong>&lt;sub&gt;τ−1&lt;/sub&gt;</td>
<td>% demand shortfall of bidder in previous month’s auction</td>
</tr>
<tr>
<td><strong>PREYIELD</strong></td>
<td>Yield of 3-month T-bill in secondary market day before auction</td>
</tr>
<tr>
<td><strong>PRESPREAD</strong></td>
<td>Yield spread of 3 month T-bill in secondary market day before auction</td>
</tr>
<tr>
<td><strong>NBIDDERS</strong></td>
<td>Number of bidders in the auction</td>
</tr>
<tr>
<td><strong>INFLATION</strong></td>
<td>Monthly CPI</td>
</tr>
<tr>
<td><strong>USD</strong></td>
<td>USD/TL exchange rate day before auction</td>
</tr>
<tr>
<td><strong>USDVOL</strong></td>
<td>USD/TL exchange rate volatility calculated for week preceding auction</td>
</tr>
<tr>
<td><strong>OMO</strong></td>
<td>Volume of Central Bank open market operations in week preceding auction</td>
</tr>
</tbody>
</table>

Daily secondary market data was obtained from the ISEBBM. Data on inflation, exchange rate and open market operations was obtained from the Central Bank of Turkey.

To control for changes in the auction environment, I add in price data from the secondary market and macroeconomic variables that are observable before the auction. These variables are defined in Table 4.

I then run linear regressions of the elasticity of the marginal value schedule on the independent variables defined in Table 4. The results are reported in Table 5. In the first regression, I add in a squared term for **SHORTFALL**<sub>τ−1</sub> to capture any non-linear effects. The coefficient on **SHORTFALL**<sub>τ−1</sub> is negative and significant at the 87.4% level. The sign of this coefficient is consistent with the hypothesis given above. If I focus on bidders who had a demand shortfall of more than 70/%, I find a more significant negative coefficient (the result is similar for higher shortfall levels).

Other significant determinants of demand elasticity are: the secondary market price observed before the auction, inflation and the exchange rate. Variables measuring uncertainty do not enter the elasticity significantly.

Although the statistical result is not very strong, the finding does have some bite regarding the operation of financial markets in developing countries like Turkey: regulatory constraints such as liquid asset reserve requirements play an important role in price formation in financial markets, a point overlooked by many finance researchers who take perfectly elastic demands for securities as given. Further investigation of this issue with

---

36 If we see K price quantity pairs given by the bidder, I average over the K – 1 elasticities implied by these discrete points using the formula η = 1/ K ∑<sub>k=1</sub> K – 1 [yk – yk+1/yk – yk+1 ∆yk]. I restrict my sample to bidders who submitted K ≥ 2 price quantity pairs in the auction.
Table 5: Regression analysis of estimated marginal valuations

<table>
<thead>
<tr>
<th>Dependent Variable</th>
<th>ELASTICITY</th>
<th>Specification</th>
<th>Full Sample</th>
<th>% Dem. Shortfall ≥ 0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>% Demand shortfall in previous auction</td>
<td>-261.12</td>
<td>-316.43</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{SHORTFALL}_{t-1}$</td>
<td>(-1.531)</td>
<td>(-1.895)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(% Demand Shortfall)$^2$</td>
<td>248.69</td>
<td>N/A$^a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\text{SHORTFALL}_{t-1}^2$)</td>
<td>(1.498)</td>
<td></td>
</tr>
<tr>
<td>Secondary market yield</td>
<td>1592.71*</td>
<td>(PREYIELD)</td>
<td>(3.867)</td>
<td>(0.145)</td>
</tr>
<tr>
<td>Secondary market yield spread</td>
<td>-53.30</td>
<td>(PREYIELD)</td>
<td>(-0.589)</td>
<td>(0.494)</td>
</tr>
<tr>
<td>Number of bidders</td>
<td>1.861</td>
<td>(NBIDDERS)</td>
<td>(1.519)</td>
<td>(0.920)</td>
</tr>
<tr>
<td>Inflation</td>
<td>20.152*</td>
<td>(INFLATION)</td>
<td>(3.810)</td>
<td>(3.508)</td>
</tr>
<tr>
<td>Dollar/TL exchange rate</td>
<td>0.466*</td>
<td>(USD)</td>
<td>(2.893)</td>
<td>(1.968)</td>
</tr>
<tr>
<td>Dollar/TL volatility</td>
<td>-0.055</td>
<td>(USDVOL)</td>
<td>(-0.098)</td>
<td>(0.332)</td>
</tr>
<tr>
<td>Volume of open market operations</td>
<td>0.0093</td>
<td>(OMO)</td>
<td>(1.677)</td>
<td>(2.453)</td>
</tr>
<tr>
<td>Constant</td>
<td>-3162*</td>
<td></td>
<td>(-4.698)</td>
<td>(-1.539)</td>
</tr>
<tr>
<td>No. of observations</td>
<td>775</td>
<td></td>
<td>258</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.0588</td>
<td></td>
<td>0.2078</td>
<td></td>
</tr>
</tbody>
</table>

T-statistics in parentheses. Variables significant at the 95% level are marked with an asterisk.

$^a$Since 169 out of the 258 observations had $\text{SHORTFALL}_{t-1}$ equal to 1, adding the squared term introduced collinearity, therefore I omit it from this regression.
much more complete data on how reserve requirements enter into a bank’s demand function for securities can shed a lot of light as to how such regulatory constraints affect market outcomes.

6 Robustness of Model Assumptions and Extensions

In this section I discuss the robustness of the empirical results if some of the model assumptions are relaxed. In particular, I investigate the effect of a common value specification that arises through the presence of resale incentives – a familiar assumption used in modelling Treasury auctions and other financial markets. Then I extend the empirical procedure to account for uncertain supply. I also outline a way to account for certain kinds of asymmetries and random participation decisions.

6.1 Common values and resale

Since resale is an important objective of the bidders in a Treasury bill auction, the assumption of private values may seem unduly restrictive. However, I will argue in this section that allowing for common values do not take away from the policy implication of the results of the previous section.

In auction theory, the presence of common values is often modeled by allowing a bidder’s valuation to depend on the signals of other bidders: $v_i(\cdot) = v_i(q, s_i, s_{-i})$ where $s_{-i}$ stands for all signals except for bidder $i$’s. Often the dependence is thought to be in the form of a statistic of the other bidders’ signals, e.g. the mean $\frac{1}{N} \sum_{j \neq i} s_j$.\footnote{This specification of a common value is also known in the literature as an “almost-common-value,” since bidders’ signals are restricted to be independent.}

From a resale perspective, this specification makes sense, as the resale price of the good in a competitive resale market can be thought of as a statistic aggregating the private information of the bidders. In my specification of common values, I will allow the ex-post marginal valuation of a bidder to depend on the realized market clearing price $p^c$, i.e.: $v_i(\cdot) = v(q, s_i, p^c)$. The market clearing price is announced by the Treasury and is observed by all bidders after the auction, and it is a statistic that aggregates the private signals of all bidders. As reported by several market participants, it also provides a benchmark for trading in the secondary market.

I will also assume that $v(q, s_i, p^1) \geq v(q, s_i, p^2)$ if $p^1 \geq p^2$ i.e. the marginal value from winning $q$ units of Treasury bills is greater if the market clearing price turns out to be high as opposed to low. This assumption also makes sense from the resale perspective, as the market clearing price serve as an indication of resale prospects. Another interpretation of this assumption can be given in terms of the “winner’s curse:” a bidder is happier to have won a close race in which the market clearing price was high than to have won by large margin in a race where the market clearing price was low.
The analysis of this addition is analyzed in detail in Appendix A. The following proposition summarizes the result:

**Proposition 3.** If a common value component of the above form exists, the marginal value estimates obtained without accounting for the common value component yields an upper bound to what the bids would have been in a Vickrey auction.

*Proof.* In the Appendix.

The intuition for this result is a familiar one from auction theory: in a private value Vickrey auction, bidders reveal their true marginal valuations, just like in a second price auction. In the presence of a common value component, bidders lower their bids to avoid the “winner’s curse.”

Hence, in the presence of resale incentives driving a common value component in ex-post utilities, the revenue losses reported in Table 3 would be a *lower bound* to the revenue losses that could be expected from switching to a Vickrey auction.

What about the comparison to a uniform price auction? From Figure 2, we see that the auctioneer’s revenue is greater in a uniform price than in a Vickrey auction if both mechanisms lead to true revelation of marginal valuations. However, in the presence of common values, “truthful revelation” loses some of its meaning in the private values case: bidders’ valuations are determined ex-post, hence an ex-ante revelation concept has to be introduced. The Vickrey auction, being the true analogue of the second price auction in the divisible good case, comes closer to such an ex-ante revelation – in spite of which, according to the empirical results of the previous section, it is revenue dominated by the discriminatory price auction in the Turkish case.

The characterization of bidding strategies and revenue performance in the uniform price auction in the common values case is still an open question from the perspective of this study. However, given that recent theory literature has pointed out that bidders can “shade” their bids quite significantly in a uniform price auction, there is reason to believe that bids in a common value uniform price auction may not be much higher than bids in a Vickrey auction.38

### 6.2 Effect of Supply Uncertainty

The results reported in the previous section do not take into account any supply uncertainty that bidders may be facing. However, the first 17 auctions in my data set are in a period in which the Treasury did not preannounce the total quantity of T-bills for sale. I will argue that the manner in which supply uncertainty is modeled has important effects to the estimation framework.

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38See, for example, Ausubel and Cramton (1997), and especially, Back and Zender (1993) and McAdams (1999), who construct equilibria of the uniform price auction game in which the auctioneer’s revenue can be zero!
Table 6: AR(1) model of Treasury bill supply

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>Coef.</th>
<th>Std. Err.</th>
<th>t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>QSOLD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>QSOLD(-1)</td>
<td>1.088</td>
<td>.140</td>
<td>7.774</td>
</tr>
<tr>
<td>CONSTANT</td>
<td>195.88</td>
<td>416.34</td>
<td>0.470</td>
</tr>
<tr>
<td>No. of observations</td>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R-squared</td>
<td>0.8119</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One way to model supply uncertainty is to view the Treasury as a strategic actor just like the bidders. In this case, the Treasury becomes a first-degree price-discriminating monopolist facing the “revealed” aggregate demand curve comprised of individual bids. Since the Treasury’s revenue is the area under the aggregate demand curve, the demand curve becomes the marginal revenue curve and the profit maximizing quantity is at the point where the aggregate demand equals marginal cost.

Another way to incorporate supply uncertainty by staying closer to the estimation framework developed in this paper is to assume that the Treasury has limited discretion in setting the supply of bills to be allocated. Discretion is limited because the Treasury has to meet the requirements of its repayment calendar. Bidders have information about the targeted supply of debt in each auction through contacts with Treasury officials (my interviews indicate this is the case), but there is residual uncertainty about the exact amount of supply.

In this case, the total supply of T-bills in auction $t$, $Q_t$, becomes a random variable which shifts the residual supply faced by each bidder and enters into the estimation of marginal valuations. The estimation method now has to incorporate this additional random variable, which I model as having a time series structure. After some specification search, I fit an AR(1) process for $QSOLD$, the quantity of T-bills sold in auction. Alternative specifications for the supply process, including those with additional variables to proxy for the borrowing requirement of the Treasury do not yield more informative results.

For the 17 auctions under the supply uncertainty regime, instead of treating the total supply as being deterministic, I used the estimated AR(1) specification of $Q_t$ to generate random draws for the total supply, and used these draws in my resampling procedure. The results, which I do not report in detail here, are very close to those reported in Table 3: the average ex-post revenue loss of the uniform price auction is 2.22% of the total revenue for the 16 auctions that were conducted in this regime. Ex-ante revenue comparisons still favor the discriminatory price auction, but not in a statistically significant fashion. Hence I conclude that the presence of supply uncertainty, when modeled as an exogenous as opposed to a strategic decision by the Treasury, does not affect the qualitative nature of my results.
The assumption that bidders are ex-ante symmetric plays a pivotal role in the resampling procedure. However, the data suggests that there can be potentially important asymmetries among bidders. Fortunately, certain kinds of asymmetries are easy to cope with in the resampling framework. If bidders can be grouped into $K \ll N$ classes, where bidders are ex-ante symmetric within a class, the resampling procedure can be modified to take separate resamples from each class.

Another closely related relaxation of model assumptions is to allow the number of bidders to be a random variable. If a stochastic process for $N$ is specified, the resampling procedure can be modified to create resamples of differing sizes, where the resample size is governed by draws from the distribution of $N$.

To incorporate asymmetry and stochastic participation into my estimation method, I divide the bidders into three categories. Type 1 bidders are those who participated in more than 21 auctions. Type 2 bidders were those who participated in 11 to 20 auctions. Type 3 bidders were those who participated in less than 10 auctions. In terms of market shares, this categorization was also quite similar. Summary statistics for participation from different categories is given in Table 7.

When simulating the residual supply function using samplings from the bid functions, I first divided the bid functions into the categories they belonged to. Then I generated three random variables, $N_1, N_2, N_3$ drawn from normal distributions with means and standard deviations given in Table 7 (and rounded to the next integer). To form my sample of bid functions for the residual supply function, I then drew $N_1, N_2, N_3$ bid functions from their respective categories.

Results under this set of modifications do not change the implication of the results in Table 3. I find that with the addition of asymmetries and stochastic participation, the estimated average ex-post revenue loss from a switch to the uniform price auction would be 4.5% of auction revenue. Once again, we see ex-ante revenue losses from the switch, but they are only significant at the 80% level.

<table>
<thead>
<tr>
<th>Category</th>
<th>Mean no. from category</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>19.65</td>
<td>4.92</td>
</tr>
<tr>
<td>Type 2</td>
<td>33.55</td>
<td>11.16</td>
</tr>
<tr>
<td>Type 3</td>
<td>12.51</td>
<td>10.04</td>
</tr>
</tbody>
</table>
7 Conclusion

On October 27, 1998, then-Deputy Secretary of the Treasury, Lawrence Summers, was quoted by the Wall Street Journal as saying “Uniform price auctions can allow the Treasury to make improvements in the efficiency of market operations and reduce the costs of financing the federal debt.” The U.S. Treasury then moved on to adopt the uniform price auction in its sale of 10-year notes and 30-year bonds. Conversely, policymakers for California’s deregulated electricity markets have recently begun debating a shift from the uniform price mechanism to the discriminatory price mechanism, following a decision in England to abandon the uniform price mechanism; attesting to the fact that the question of mechanism choice in divisible good auctions is still very much alive.39

The empirical results of this paper seem to counter the policy decision to abandon the discriminatory price auction in favor of the uniform price or Vickrey auction. However, the discussion in Section 5.2 suggests that this result may be due to the special circumstances Turkish banks face in trying to meet regularity constraints on their liquid asset reserves. Banks facing binding reserve requirements can rationalize placing high bids on units that are almost certain to win. These high bids are captured as revenue in a discriminatory auction, but not in a uniform price or Vickrey auction.

From an auction theoretic point of view, the above explanation may be subject to scrutiny in light of the “revenue equivalence theorem” of Myerson (1981). According to the revenue equivalence theorem, extended by Krishna and Perry (1999) to the multi-unit auction setting, in an independent private values environment, mechanisms that yield an efficient outcome should yield the same expected revenue. However, a calculation using my estimates of bidders’ marginal valuations reveals that, on average, the discriminatory price auction causes an efficiency loss that amounts to 11% of total surplus obtainable from the auction. This means that the discriminatory mechanism incented some bidders with high marginal values to bid lower than bidders with higher marginal values, and vice versa.40

In contrast, the Vickrey mechanism would achieve an efficient outcome in an independent private values environment, achieving the first objective of the quote by Summers. However, as pointed out in many auction theoretic results, efficiency and revenue extraction can sometimes be conflicting objectives, a result that is corroborated in this study.41

40 Observe that the first-order condition that I use to identify the marginal valuations is not subject to a “single-crossing” condition that would guarantee a sorting of the bid vectors in some (lattice) ordering of the marginal valuation functions of the bidders. Therefore, efficient outcomes are not guaranteed by the use of first-order conditions per se. A recent generalization of the “single-crossing” conditions of Athey (2000) to the multi-unit auction case is given by McAdams (2000). I am currently exploring the impact of such single-crossing conditions on the characterization of equilibrium bids in the discriminatory auction.
41 Myerson (1981) was first to show this in the optimal auction context, observing that the distribution of valuations needs to satisfy a monotone hazard ratio property for standard auctions to be optimal from a revenue maximization viewpoint. Ausubel and Cramton (1997) and Ausubel and Cramton (1998a) discuss the extension of this result in the multi-unit context.
In the discriminatory vs. uniform price or Vickrey debate, many proponents of uniform price or Vickrey auctions, including Friedman (1960), have suggested that more bidders are likely to participate in a uniform price or Vickrey auction, causing more efficient or potentially more lucrative auction outcomes, since strategizing becomes less of an issue. This suggestion is not very precise, since although truthful revelation of marginal valuations is a weakly dominant strategy in the Vickrey auction, the same cannot be said about the uniform price auction, since Ausubel and Cramton (1997) show that bidders typically have an incentive to shade their bids in these auctions.42

The comparison of the effect of an increased number of bidders across discriminatory price and uniform price auction is an issue that has not been explored theoretically. To analyze the effect of increased bidder participation on auction outcomes, the empirical approach utilized here has to be modified to explicitly solve for the equilibrium of an auction with additional bidders. The arguments leading to the empirical approach utilized here does provide some insight into how the equilibrium of the game can be solved for, as discussed in Section 4. I believe that this is a promising direction for future research in this area.

A very promising area of future research is to look at the effect of debt supply decisions by the Treasury in a dynamic context. Especially in an environment with liquid asset reserve requirements, banks will structure their bidding decisions across different auction by taking the future decisions of the Treasury into account. By analyzing the response of bidders to the actions of the Treasury, a careful characterization of “optimal debt-servicing rules” can be obtained.

The analysis and estimation method employed here can easily be carried over to different auction settings to conduct the policy experiments reported in this paper. Preliminary results communicated by Leonardo Rezende suggest that significant revenue losses would have been incurred by the Brazilian Treasury had the authorities adopted a uniform price auction scheme. Data from auctions conducted in recently deregulated power exchanges around the world can also be analyzed to answer similar questions about mechanism choice.43 I should note that in most power exchanges, a uniform price auction format is used. However, the optimality condition that drives the estimation procedure here can easily be derived for the uniform price auction, and the procedure can be repeated. Another market that can be analyzed is the national exchange for sulfur-dioxide emission permits which uses a discriminatory auction format (Joskow, Schmalensee and Bailey (1998)). The method developed in this paper can be used to investigate the extent of the exercise of market power in this exchange, as a relatively small number of bidders (e.g. Enron) account for a large fraction of the bids.44

42Wilson (1979), Back and Zender (1993), and McAdams (1999) also provide examples in which bidders shade their bids in the uniform price auction, yielding some outcomes in which the auctioneers’ revenue is very low.
44There are two main complications in replicating the methods developed here to analyze the sulfur-dioxide emissions market. The first is to account for private offers to sell, as supply decisions are made by private parties, who presumably also act strategically from the bidders’ point of view. The other complication is that, based on the analysis of bids and offers by Joskow et al. (1998), there is reason to think that these auctions are closer to a pure common value auction than a private value auction. The empirical
A Proofs of propositions in the text

A.1 Proof of Proposition 1

Now, observe that after an interchange of summations, and the addition of the monotonicity constraints, the Lagrangian of the objective function of the bidder given in equation (3) can be written as:

\[ \mathcal{L} = \sum_{k=1}^{K} H(p_k, \bar{y}'_k) \left( \int_{y_{ik+1}}^{y_{ik}} v_i(q, s_i) dq - p_k (y_{ik} - y_{ik+1}) \right) + \lambda_k (y_{ik} - y_{ik+1}) \]  

(12)

Given the above restrictions on \( v(\cdot) \), the first-order conditions for a maximum are, for each \( k \) (observe that the \( \varepsilon \) in the statement of the proposition is a technical device to allow for differentiation under the integral sign):

\[ H(p_k, \bar{y}'_k)[v_i(y_{ik}, s_i) - p_k] + \frac{\partial H(p_k, \bar{y}'_k)}{\partial y_{ik}} \left( \int_{y_{ik+1}}^{y_{ik}} v_i(q, s_i) dq - p_k (y_{ik} - y_{ik+1}) \right) + \lambda_k = 0 \]

(13)

In reality, I see bids at only a subset of the possible price points. Let us denote the observed bids as in Proposition 1: \( \{y_{ik} > y_{ik+1} > ... > y_{ik+2}\} \). I will interpret these observed bids as being at price points at which the monotonicity constraint \( y_{ik} \geq y_{ik+1} \) is not binding, i.e. \( \lambda_k = 0 \). For all other \( k, \lambda_k > 0 \).

To ease notation for the rest of the analysis, define:

\[ A_{km+1} = \frac{\partial H(p_{km+1}, \bar{y}'_m)}{\partial y_{km+1}} \left( \int_{y_{km+2}}^{y_{km+1}} v_i(q, s_i) dq - p_{km+1} (y_{km+1} - y_{km+2}) \right) \]

and

\[ B_{km+1} = \frac{\partial H(p_{km}, \bar{y}'_m)}{\partial y_{km+1}} \left( \int_{y_{km+1}}^{y_{km+2}} v_i(q, s_i) dq - p_{km} (y_{km+2} - y_{km+1}) \right) \]

Now, write the first-order conditions for all price points \( j \) such that \( k^m + 1 \leq j \leq k^{m+1} \), I get

\[ H(p_{km+1}, \bar{y}'_m)[v_i(y_{km+1}, s_i) - p_{km+1}] + A_{km+1} + \lambda_{km+1} = H(p_{km}, \bar{y}'_m)[v_i(y_{km+1}, s_i) - p_{km}] + B_{km+1} \]

and going all the way to \( k^{m+1} \):

\[ H(p_{km+1}, \bar{y}'_m)[v_i(y_{km+1}, s_i) - p_{km}] + A_{km+1} = H(p_{km+1-1}, \bar{y}'_m)[v_i(y_{km+1}, s_i) - p_{km+1-1}] + B_{km+1} + \lambda_{km+1-1} \]

Adding these first-order conditions from \( k^m + 1 \leq j \leq k^{m+1} \), we see that the Lagrange multipliers conveniently cancel out from successive equations and the overall sum telescopes. Since for \( k^m + 1 \leq j < k^{m+1} \), bound obtained in section 6.1 would still apply, but it is desirable to obtain a modification of the empirical procedure to account more explicitly for the existence of a common value.
y_{ij} = y_{ij+1}$, the integral terms vanish in $A_{k^L}$ and $B_{k^L}$ for such $j$. Also, what is left of the intermediate terms in for these intermediate $A_{k^j}$ and $B_{k^j}$ cancel out, since $\frac{\partial H}{\partial y_{ij}}$ is the same across consecutive $A_{k^j}$ and $B_{k^j}$ terms. Hence, we are left with:

$$H(p_{k^m+1}, \bar{y}^L)\{v_i(y_{ik^m+1}, s_i) - p_{k^m+1}\} + A_{k^m+1} = H(p_{k^m}, \bar{y}^L)\{v_i(y_{ik^m+1}, s_i) - p_{k^m}\} + B_{k^m+1}$$

where I have used the fact that $y_{ik^m+1} = y_{ik^m+1}$, $y_{ik^m+1+1} = y_{ik^m+2}$.

Observe that this is an expression entirely in terms of the observed bids $\{y_{ik^1} > y_{ik^2} > ... > y_{ik^L}\}$.

Rewriting this to solve for $v_i(y_{ik^m+1}, s_i)$, we get:

$$v_i(y_{ik^m+1}, s_i) = p_{k^m+1} + \frac{H(p_{k^m}, \bar{y}^L)[p_{k^m+1} - p_{k^m}]}{H(p_{k^m+1}, \bar{y}^L) - H(p_{k^m}, \bar{y}^L)} + \frac{B_{k^m+1} - A_{k^m+1}}{H(p_{k^m+1}, \bar{y}^L) - H(p_{k^m}, \bar{y}^L)}$$

Now, with the restrictions stated in the proposition about the marginal valuation function, the integrals in $A_{k^m+1}$ and $B_{k^m+1}$ can be evaluated and these terms become:

$$A_{k^m+1} = \frac{\partial H(p_{k^m+1}, \bar{y}^L)}{\partial y_{ik^m+1}}\{[v_i(y_{ik^m+2}, s_i) - p_{k^m+1}](y_{ik^m+1} - y_{ik^m+2})\} + O(\varepsilon)$$

$$B_{k^m+1} = \frac{\partial H(p_{k^m}, \bar{y}^L)}{\partial y_{ik^m+1}}\{[v_i(y_{ik^m+1}, s_i) - p_{k^m}](y_{ik^m} - y_{ik^m+1})\} + O(\varepsilon)$$

Since we can make $\varepsilon$ arbitrarily small, I will ignore the $O(\varepsilon)$ terms.

To analyze the “boundary condition” for the bid with the lowest price, $y_{ik^1}$: this bid implies that the bidder is also willing to accept $y_{ik^1}$ units of the security at the minimum price on the price grid, $p_0$. I will assume that this implied bid is also an “observed bid,” where $y_{i0} = y_{ik^1}$. Therefore, the first-order relation in equation (A.1) applies. But, by the restriction on the marginal valuations, the integral in $B_{k^1}$ vanishes up to a term in $\varepsilon$, and since $y_{i0} = 0$, $B_{k^1} = p_0y_{ik^1}$. Since bidders can submit any positive price, I can safely set $p_0 = 0$, so this makes $B_1 = 0$. As for $A_{k^1}$, we can calculate this like the other $A_{k^j}$’s, so $A_{k^1} = \frac{\partial H(p_{k^1}, \bar{y}^L)}{\partial y_{ik^1}}\{[v_i(y_{ik^2}, s_i) - p_{k^1}](y_{ik^1} - y_{ik^2})\}$.

Also, at this price, $H(p_0, \bar{y}^L) = 0$ by the assumption in the model setup. So:

$$v_i(y_{ik^1}, s_i) = p_{k^1} - \frac{\partial H(p_{k^1}, \bar{y}^L)}{\partial y_{ik^1}}\{[v_i(y_{ik^2}, s_i) - p_{k^1}](y_{ik^1} - y_{ik^2})\}$$

Although this condition appears as if the bidder is bidding a price below his marginal valuation, observe that $\frac{\partial H(p_{k^1}, \bar{y}^L)}{\partial y_{ik^1}} \leq 0$ by the following intuitive argument: for every upward sloping residual supply curve that the bidder expects to face, increasing his quantity demand at a particular price point either increases the market clearing price or leaves it the same. Hence, the probability that the market clearing price is below a given price becomes less or equal than what it was.

As for the other “boundary condition” for the bid with the highest price, $y_{ik^L}$: this bid implies that the bidder is not willing to accept any units of the security for prices above $p_{k^L}$. Hence, the “implied bid” for price $p_{k^L+1}$ is $y_{ik^{L+1}} = 0$. 

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\[ v_i(y_{ik^L}, s_i) = p_{k^L} + \frac{H(p_{k^{L-1}}, \bar{y}_{i}^L)}{H(p_{k^L}, \bar{y}_{i}^L)}[p_{k^L} - p_{k^{L-1}}] + \frac{B_{k^L} - A_{k^L}}{H(p_{k^L}, \bar{y}_{i}^L) - H(p_{k^{L-1}}, \bar{y}_{i}^L)} \]

where

\[ A_{k^L} = -\frac{\partial H(p_{k^L}, \bar{y}_{i}^L)}{\partial y_{ik^L}} p_{k^L} y_{ik^L} \]

since \( v_i(y_{ik^L+1}, s_i) = v_i(0, s_i) = 0 \). Also:

\[ B_{k^L} = \frac{\partial H(p_{k^{L-1}}, \bar{y}_{i}^L)}{\partial y_{ik^L}} \left\{ [v_i(y_{ik^L}, s_i) - p_{k^{L-1}}(y_{ik^L-1} - y_{ik^L})] \right\} \]

Hence, given estimates of \( H(p_k^*, \bar{y}_{i}^L) \) and \( \frac{\partial H(p_k^*, \bar{y}_{i}^L)}{\partial y_{ik^L}} \), we have a recursive set of \( L \) linear equations in \( v_i(y_{ik^L}, s_i), \ldots, v_i(y_{ik^L}, s^L) \) which we can solve for the marginal valuations.

As for the second-order conditions: if \( \frac{\partial^2 H}{\partial y_{ik^L} \partial y_{ik^L}} = 0 \), then restricting \( v_i(y, s_i) \) to be a decreasing function of \( y \) guarantees sufficiency. This assumption says that changing one’s quantity bid does not shift the distribution of the market clearing price – which is the assumption in Nautz (1995). However, if a bidder commands a large fraction of the total demand, he will take into account the fact that the probability distribution of the market clearing price will shift with his bid. In this case, we need the additional conditions \( \frac{\partial H}{\partial y_{ik^L}} \leq 0 \) and \( \frac{\partial^2 H}{\partial y_{ik^L} \partial y_{ik^L}} \leq 0 \) to hold. The first condition is automatically satisfied in the model, since for every upward sloping residual supply curve that the bidder expects to face, increasing his quantity demand at a particular price point either increases the market clearing price or leaves it the same. Hence, the probability that the market clearing price is below a given price becomes less or equal than what it was. The second condition does not have a natural justification, therefore I will treat it as an additional assumption that could potentially be empirically tested.

### A.2 Proof of Lemma 1

**Lemma.** \( H(p_k, \bar{y}_{i}^L) = \Pr \{ p_{k^*} \leq p_k | \bar{y}_{i}^L \} = \Pr \left\{ \frac{1}{N-1} \sum_{j \neq i}^N y_k(s_j) \leq \frac{Q - y_k(s_i)}{N-1} | s_i \} \right. \). That is, the probability distribution of the market clearing price can be represented as the probability distribution of the sample mean of \( N - 1 \) i.i.d. random variables, \( \{y_k(s_1), \ldots, y_k(s_{i-1}), y_k(s_{i+1}), \ldots, y_k(s_N)\} \).

**Proof.**

Since \( k^* = \min \{ k : \sum_{i=1}^N y_k(s_i) \leq Q \} \), \( I \{ p_k \leq p_{k^*} \} = I \{ \sum_{i=1}^N y_k(s_i) > Q \} \), i.e. for any price strictly below the market clearing price, aggregate demand will strictly exceed supply. Taking the complement of this event, we get \( I \{ p_k \leq p_{k^*} \} = I \{ \sum_{i=1}^N y_k(s_i) \leq Q \} \). Note that implicit in this argument is that \( y_k \)’s are declining in \( k \).
A.3 Proof of Proposition 2

Proposition. Suppose we observe $L$ bid vectors $\{\vec{y}(s_1), \cdots, \vec{y}(s_L)\}$ generated by $L$ i.i.d. draws from the distribution of private bidder signals, where $y(s)$ denotes the symmetric pure strategy equilibrium of the discriminatory price auction with $N$ bidders. Then, as $L \to \infty$, $\hat{H}^R(p_k, \vec{y}_i)$ converges to $H(p_k, \vec{y}_i)$ almost surely.

Proof.

I will use the representation of $H(p_k, \vec{y}_i)$ in Lemma 1. To simplify notation, I shall drop the dependence of the quantity bids on the signals, drop the component subscript, and refer to my data as the sample: $\{Y_1, \cdots, Y_{N-1}\}$. Let $F(y)$ denote the probability distribution function that generated $\{Y_1, \cdots, Y_{N-1}\}$, and in equilibrium, is known to the bidder. I will also replace the quantity $\frac{Q - y_k(s_i)}{N-1}$ by the constant $c$. Also, let $\bar{Y}_{N-1} = \frac{1}{N-1} \sum_{j=1}^{N-1} Y_j$, i.e. the $(N-1)$ fold sample mean. Then

$$H(p_k, \vec{y}_i) = \Pr(\bar{Y}_{N-1} \leq c) \quad (14)$$

i.e. the distribution function of the sample mean of $\{Y_1, \cdots, Y_{N-1}\}$. Now develop the “resampling” estimator for $H(p_k, \vec{y}_i)$ in the following way: Suppose we see $L$ independent draws from the distribution $F(y)$, giving us the random variables $\{Y_1, \cdots, Y_L\}$. This occurs, for example, if we see $l$ repetitions of the same auction, with the signals of the bidders drawn differently each time, giving us $L = lN$ data points in total. Draw a sample of $(N-1)$ variables with replacement from $\{Y_1, \cdots, Y_L\}$ with equal probability $\frac{1}{L}$ given to each sample point. Call this sample $\{Y^*_1, \cdots, Y^*_{N-1}\}$. Then:

$$\Pr(Y^*_1 \leq y_1, \cdots, Y^*_{N-1} \leq y_{N-1}) = F_L(y_1) \cdots F_L(y_{N-1}) \quad (15)$$

where

$$F_L(y_1) = \frac{1}{L} \sum_{i=1}^{L} I(Y_i \leq y_1) \quad (16)$$

i.e. the empirical distribution function generated by the sample $\{Y_1, \cdots, Y_L\}$.

Let $\bar{Y'}_{N-1} = \frac{1}{N-1} \sum_{j=1}^{N-1} Y^*_j$. The resampling estimator of $H(p_k, \vec{y}_i) = \Pr(\bar{Y'}_{N-1} \leq c)$ is given by:

$$\hat{H}^R(p_k, \vec{y}_i) = \Pr(\bar{Y'}_{N-1} \leq c) \quad (17)$$

To establish consistency, we would like to show the following:

Lemma 2. $\lim_{L \to \infty} \Pr(\bar{Y'}_{N-1} \leq c) = \Pr(\bar{Y}_{N-1} \leq c)$ for all $c$ if $F(y)$ is continuous.

Proof. Start with the conditional probability distribution function $\Pr(\bar{Y}_{N-1} \leq c | Y_1, \cdots, Y_L)$. Define the condi-
tional characteristic function\(^5\) corresponding to this distribution function as:

\[ E \left[ \exp(it\overline{Y}_{N-1}) | Y_1, \ldots, Y_L \right] = (E_F \left[ \exp(it\overline{Y}_{N-1}) \right])^{N-1} \]

\[ = \left( \int \exp(ity) dF(y) \right)^{N-1} \] (19)

By the Glivenko-Cantelli Theorem, \( \lim_{L \to \infty} F_L(y) \overset{a.s.}{=} F(y) \), therefore \( \lim_{L \to \infty} \int \exp(ity) dF_L(y) \overset{a.s.}{=} \int \exp(ity) dF(y) \) at the points of continuity of \( F \). Hence:

\[ \lim_{L \to \infty} E \left[ \exp(it\overline{Y}_{N-1}) | Y_1, \ldots, Y_L \right] \overset{a.s.}{=} E \left[ \exp(it\overline{Y}_{N-1}) \right] \] (20)

By Levy’s convergence theorem, this implies that \( \lim_{L \to \infty} \Pr\{\overline{Y}_{N-1} \leq c | Y_1, \ldots, Y_L \} \overset{a.s.}{=} \Pr\{\overline{Y}_{N-1} \leq c \} \) if \( F(y) \) is continuous. Now \( \Pr\{\overline{Y}_{N-1} \leq c | Y_1, \ldots, Y_L \} \) is bounded by 1, so by the bounded convergence theorem:

\[ \lim_{L \to \infty} \Pr\{\overline{Y}_{N-1} \leq c \} \to \int_{\Omega} \lim_{L \to \infty} \Pr\{\overline{Y}_{N-1} \leq c | Y_1, \ldots, Y_L \} d\omega = \Pr\{\overline{Y}_{N-1} \leq c \} \] (21)

The lemma establishes the proposition.

An alternative approach is to regard \( H(p_k, \overline{y}_l) \), as a V-statistic and define:

\[ \hat{H}^V(p_k, \overline{y}_l) = \frac{1}{L^{N-1}} \sum_{d=1}^{L} \sum_{d^N=1}^{L} I\{ \frac{1}{N-1} \sum_{j=1}^{N-1} Y_{d^N} \leq c \} \] (22)

where the indicator variable is evaluated over all \( L^{N-1} \) unique \((N-1)\)-tuples of the \( L \) data points; instead of over random draws.

The literature on V-statistics provides the following asymptotic result:

**Proposition.** \( \hat{H}^V(p_k, \overline{y}_l) \) is a consistent estimator of \( H(p_k, \overline{y}_l) \) as \( L \to \infty \).

**Proof.** Define \( \sigma^2 = \text{Cov}(\frac{1}{N-1}(Y_1 + \ldots + Y_j + Y_{j+1} + \ldots + Y_{N-1}) \leq c), I\{ \frac{1}{N-1}(Y_1 + \ldots + Y_j + Y_{j+1} + \ldots + Y_{N-1}) \leq c \} \). The covariance of two indicator variables is bounded in absolute value by 1, hence \( \sigma^2 < \infty \) for all \( j \). Also, \( E[\hat{H}^V(p_k, \overline{y}_l)] = H(p_k, \overline{y}_l) \), hence the estimator is unbiased. Hence, by theorem 6.2.2. in Lehmann (1999):

\[ \lim_{L \to \infty} \text{Var}(\frac{1}{L} \hat{H}^V(p_k, \overline{y}_l)) = (N - 1)^2 \sigma^2_1 \]

\[ \sqrt{L}(\hat{H}^V(p_k, \overline{y}_l) - H(p_k, \overline{y}_l)) \overset{D}{\to} N(0, (N - 1)^2 \sigma^2_1) \]

and

\[ \frac{\hat{H}^V(p_k, \overline{y}_l) - H(p_k, \overline{y}_l)}{\text{Var}(\hat{H}^V(p_k, \overline{y}_l))} \overset{D}{\to} N(0, 1) \]

\(^{45}\) I thank Evarist Giné for suggesting the use of characteristic functions.
In the practical application considered in the paper, we have the case that \( L = N \). The asymptotic variance of the estimates will be \( \frac{N^2}{L} \sigma_1^2 = \frac{N^2}{N} \sigma_1^2 \), where \( \sigma_1^2 = \text{Var}(I\{\frac{1}{N-1}(y_1 + \ldots + y_j + y_{j+1} + \ldots + y_{N-1}) \leq c\}) \). This variance is bounded above by 1. Since \( \frac{(N)^2}{N} \approx 70 \) in practice, and \( H(p_k, \bar{y}_i) \) is a probability, \( \sigma_1^2 \) has to be a small number for the asymptotic variance to be small. However, since \( L \) is near \( N \), the asymptotic variance formula obtained above might not be the correct formula to use, since its proof relies on the fact that \( L \to \infty \) and that \( N \) stays fixed.

One issue is that \( \hat{H}^V(p_k, \bar{y}_i) \) as defined above is computationally infeasible, since for \( L = N = 70 \), there are \( 70^{69} \) possible draws with replacement. Hence, \( \hat{H}^R(p_k, \bar{y}_i) \) essentially approximates \( \hat{H}^V(p_k, \bar{y}_i) \) for a finite number of bootstrap draws \( B \). Since both estimates are bounded above, as \( B \) goes to infinity, \( \text{Var}(\hat{H}^R(p_k, y_i) - \hat{H}^V(p_k, \bar{y}_i)) \) goes to zero at rate \( \frac{1}{B} \), so the two estimators will be equivalent.

### A.4 Proof of Proposition 3

**Proposition.** Marginal value estimates obtained without accounting for resale incentives yield an upper bound to what the bids would have been in a Vickrey auction.

**Proof.**

It is much easier to analyze the impact of resale incentives using a version of the bidding game in which the prices are no longer constrained to be on discrete grid. This is the “share auction” model of Wilson (1979). Let \( y_i(p) : \mathbb{R} \to \mathbb{R} \) be the bid function that bidder \( i \) submits, which is constrained to be decreasing. Let \( Q \) be the amount of Treasury bills for sale. With \( N \) competing bidders, the market clearing price, \( p^c \), will be at the point where

\[
y_i(p^c) = Q - \sum_{j \neq i}^N y_j(p^c)
\]

i.e. where bidder \( i \)’s demand curve intersects his “residual supply curve.” Now, analogous to the discrete model, define:

\[
H(p, y_i(p)) = \Pr\{y_i(p) \leq Q - \sum_{j \neq i}^N y_j(p)\} = \Pr\{p^c \leq p\}
\]

which is the probability distribution function of the market clearing price. I will assume that \( H \) has a bounded, well-defined density over the support of market clearing prices. I will also assume that \( H \) is differentiable in both of its arguments. Then, for a given market clearing price, \( p^c \), the surplus of bidder \( i \) in a discriminatory auction is given by:

\[
\int_0^{y_i(p^c)} v(q, s_i, p^c) - y_i^{-1}(q) dq
\]
where the first term is the ex-post surplus a bidder gets from winning \( y_i(p^c) \) units of T-bills upon learning the market clearing price, \( p^c \). The second term is the payment of the bidder, i.e. the area under his revealed demand curve to the point it intersects the market clearing price. Since \( H(.) \) defines the probability distribution over the set of market clearing prices, the bidder’s expected profit maximization problem before observing the market clearing price is:

\[
\max_{y(\cdot)} \int_0^\infty \left( \int_0^{y_i(p^c)} v(q, s_i, p^c) - y_i^{-1}(q) dq \right) dH(p^c, y_i(p^c))
\]  

(27)

where \( dH(p, y_i(p)) \) (denoting the total derivative of \( H \)) is the probability density function of the market clearing price.

This setup can be analyzed using calculus of variations, but first we have to get rid of the inner integral. Let the profit of the bidder from submitting the bid function \( y(p) \) be

\[
\pi(y(p)) = \int_0^{y(p)} v(q, s_i, p) - y_i^{-1}(q) dq
\]  

(28)

Now, setting \( y(\infty) = 0 \):

\[
\pi(y(\infty)) = \pi(0) = 0
\]  

(29)

and

\[
\frac{d\pi}{dp} = \left[ v(y(p), s_i, p) - y_i^{-1}(y(p)) \right] y'(p) + \int_0^{y(p)} \frac{\partial}{\partial p} v(q, s_i, p) dq
\]  

(30)

\[
= \left( v(y(p), s_i, p) - y_i^{-1}(y(p)) \right) y'(p) + w(p, y(p))
\]  

(31)

Substituting in the expression for \( \pi(y(p)) \) in the objective function and integrating by parts, I get

\[
\max_{y(\cdot)} - \int_0^\infty \left( (v(y(p), s_i, p) - p) y'(p) + w(p, y(p)) \right) H(p, y(p)) dp
\]  

(32)

Observe that the integrand is a function of \( p, y \) and \( y' \), denote it by \( F( p, y, y' ) \). The Euler equation (which is a necessary condition for optimality) is given by (Kamien and Schwartz (1993)):

\[
F_y = \frac{d}{dp} F_y'
\]  

(33)

Evaluating the derivatives, I get:

\[
v(y(p), s_i, p) = p + \frac{H(p, y(p))}{H_p(p, y(p))} + w(p, y_i(p)) \frac{H_y(p, y(p))}{H_p(p, y(p))}
\]  

(34)

Following the derivation in the appendix, the necessary condition for an optimal bid function is found to be:

\[
v(y_i(p), s_i, p) = p + \frac{H(p, y_i(p))}{H_p(p, y_i(p))} + w(p, y_i(p)) \frac{H_y(p, y_i(p))}{H_p(p, y_i(p))}
\]  

(35)
where $H_p$ and $H_y$ denote the partial derivatives of $H(\cdot)$ with respect to $p$ and $y$, respectively. Here,

$$w(p, y_i(p)) = \int_0^{y_i(p)} \frac{\partial}{\partial p} v(q, s_i, p) dq$$

i.e. the marginal change in bidder $i$’s expected surplus given a change in the market clearing price.

Two important observations to be made are:

- When $\frac{\partial}{\partial p} v(q, s_i, p) = 0$

That is, when there is no resale incentive, the first order condition becomes $v(y_i(p), s_i, p) = p + \frac{H(p, y_i(p))}{H_p(p, y_i(p))}$. Observe that this is the limit of the necessary condition (12) of the private value discrete price grid game as $p_{k-1} - p_k \to 0$.

- $w(p, y_i(p)) \frac{H_y(p, y_i(p))}{H_p(p, y_i(p))} \leq 0$.

(Sketch Proof:) $H_p \geq 0$, since if $p$ increases, then the probability that the market clearing price will be below $p$ will also increase. If $y_i(p)$, i.e. my demand at price $p$, increases, then the probability that I will get $y_i(p)$ will decrease. Hence $H_y \leq 0$. Finally $w(p, y) \geq 0$ since ex-post marginal value is defined to be increasing in the market clearing price.

The second observation implies that the estimate of the marginal valuation $\hat{v}(y_{ik})$ using the “private value” necessary condition provides an upper bound to $v(y_{ik}, s_i, p^k)$, the ex-post marginal valuation in an auction with resale incentives. Furthermore, $\hat{v}(y_{ik}) \geq v(y_{ik}, s_i, p^k)$, i.e. the estimated marginal valuation is also an upper bound to the ex-post marginal valuation in an auction whose market clearing price is $p^k$.

Estimating a bound on $v(y_{ik}, s_i, p^k)$ has a useful application, since this is the bidding strategy in a Vickreya auction with resale incentives. To see this, once again look at the continuous version of the game, where the objective function of the bidder becomes:

$$\max_{y(.)} \int_0^{\infty} \left\{ \int_0^{y(.)} v(q, s_i, p^c) dq - E_s_{s-1|p^c, s_i} RS^{-1}(q, s_{-i}) dq \right\} dH(p^c, y_i(p^c))$$

where $RS_i(p, s_{-i}) = Q - \sum_{j \neq i}^N y(p, s_j)$ is the “residual supply function” that bidder $i$ faces. Recall that the payment rule in the Vickreya auction is that each bidder pays the area under his residual supply function up to the market clearing price.

Now let’s look at the quantity $y_{ik}$ a bidder would demand if she knew that the market-clearing price equalled $p^k$:

$$\max_{y_{ik}} \int_0^{y_{ik}} v(q, s_i, p^k) - E_s_{s-1|p^c, s_i} RS^{-1}(q, s_{-i}) dq$$

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Taking the first-order condition with respect to $x$, we get:

$$v(y_{ik}, s_i, p^k) = E_{s_{-i}|p^k, s_i}RS^{-1}(y_{ik}, s_{-i})$$

$$= p^k$$

i.e. the optimal choice of $y_{ik}$ is such that $v(y_{ik}, s_i, p^k) = p^k$. In other words, in a Vickrey auction, the bid pair is $\{p^k, y_{ik}\}$ is actually $\{v(y_{ik}, s_i, p^k), y_{ik}\}$.$^{46}$ Since $\hat{v}(y_{ik}) \geq v(y_{ik}, s_i, p^k)$, the marginal value estimates provide an upper bound to what the bids would have been had the auction been run as a Vickrey auction.

### B Small sample properties and standard errors

In the practical application considered in the paper, we have the case that $L = N - 1$, where $L$ refers to the number of “data points” (in terms of bid vectors) we have for the asymptotic case considered Proposition 1.$^{47}$ Observe that for $L = N - 1$, $H(p_k, \bar{y}_i) = \Pr\{p_k \leq p_k | \bar{y}_i\} = \Pr\{\frac{1}{N-1} \sum_{j \neq i} y_k(s_j) \leq \frac{Q - y_k(s_i)}{N-2} | s_i\}$ is just the standard bootstrap estimator of the distribution function of the sample mean, by Lemma 1 and the arguments in the setup of Proposition 1. The small sample properties of this pointwise estimate of the probability distribution function is going to depend on where $c = \frac{Q - y_k(s_i)}{N-1}$, the number of bootstrap draws $B$, and the number of data points $N$.

Small sample properties of bootstrap estimators are discussed in depth by the books by Hall (1992) and Efron and Tibshirani (1998). In particular, Hall (1992) shows that the bootstrap estimate of the distribution of the sample mean is, in most cases, as good or better than a standard normal approximation using the sample mean and variance. In the following section, I discuss two methods that can be used to assess the standard errors of the bootstrap estimates.

The choice of the appropriate number of bootstrap draws is discussed widely in the applied statistical literature. Efron and Tibshirani (1998) suggest that for bootstrap distribution estimators in most applied settings, $B$ should be at least 1000. I use $B = 10000$ in my application.

#### B.1 Standard errors

Recall that the estimated marginal valuation for each bid $y_{ik}$ is given by the formula:

$$\hat{v}(y_{ik}) = p_k + \frac{\hat{H}^b(p_k, \bar{y}_i)[p_k - p_{k-1}]}{\hat{H}^b(p_k, \bar{y}_i) - \hat{H}^b(p_{k-1}, \bar{y}_i)}$$

$^{46}$The analogy to the single-unit Vickrey auction (second price auction) with common values is clear: in the second price common value auction, you bid your expected value conditional on your bid being tied to win (Milgrom and Weber (1982)). In the multi-unit case, you bid your expected value conditional on your bid being at the market clearing price; that is, you bid your expected value, corrected for the “winner’s curse.”

$^{47}$We can also have $L = N$ if we add bidder $i$’s bid into the data as well – which we can do since all bidders exchangeable given the model assumptions.
where $\hat{H}^b(p_k, \vec{y}_i)$ has non-zero variance about the true probability distribution of the market clearing price, $H(p_k, \vec{y}_i)$. Unfortunately, the consistency proof does not lead the way to a asymptotic formula for this variance. Calculating the standard errors introduced into the marginal valuations is doubly difficult since it is a non-linear function in $H(.)$.

Given the complexity of the problem, I first use the “jackknife-after-bootstrap” method suggested by Efron (1992) to compute the standard errors of my estimates of the marginal valuation. Specifically, $\hat{v}_B(y_{ik})$ be the estimate for the marginal valuation of bidder $i$ for $y_{ik}$ units of Treasury bills, calculated using $B$ bootstrap simulations of the market clearing price. Then, the “jackknife-after-bootstrap” estimator of the variance of the estimated marginal valuation is defined to be:

$$\text{var}_{\text{jack}}(\hat{v}_B(y_{ik})) = \frac{N}{N-1} \sum_{j=1}^{N} (\hat{v}_B(j)(y_{ik}) - (\frac{1}{N} \sum_{i=1}^{N} \hat{v}_B(j)(y_{ik})))^2$$

(42)

where $\hat{v}_B(j)(y_{ik})$ is the bootstrap estimate of $\hat{v}_B(j)(y_{ik})$ calculated over a set of resamples that do not contain the bid vector $\vec{y}_j$. The validity of using this procedure is discussed at length in Efron (1992).

Another way to get at the standard errors is to compute the variance of $\hat{H}^b(p_k, \vec{y}_i)$ directly, and use the delta method to compute the variance of the $v(\hat{y}_{ik})$.

Making the abbreviations $\hat{H}_k = \hat{H}^R(p_k, \vec{y}_i)$ and $\hat{H}_{k-1} = \hat{H}^R(p_{k-1}, \vec{y}_i)$:

$$\text{var}(\hat{v}(y_{ik})) = \frac{(p_k - p_{k-1})^2}{(\hat{H}_k - \hat{H}_{k-1})^4} \left[ \text{var}(\hat{H}_k)\hat{H}_k^2 + \text{var}(\hat{H}_{k-1})\hat{H}_{k-1}^2 - 2\text{Cov}(\hat{H}_k, \hat{H}_{k-1})\hat{H}_k\hat{H}_{k-1} \right]$$

(43)

To get at the variance of $\hat{H}^R(p_k, \vec{y}_i)$, I use my $B$ bootstrap draws of $I\{\frac{1}{N-1} \sum_{j=1}^{N-1} Y_j^* \leq c\}$. Call each draw $X_i, i = 1..B$, which are Bernoulli random variables. Given these Bernoulli draws, the variance of $\hat{H}^b(p_k, \vec{y}_i)$ is given by:

$$\text{Var}(\hat{H}^R(p_k, \vec{y}_i)) = \frac{1}{B^2} \sum_{i=1}^{B} \text{Var}(X_i) - \frac{1}{B(B-1)} \sum_{i=1}^{B} \sum_{j \neq i} \text{Cov}(X_i, X_j)$$

(44)

Observe that the covariance terms are not zero, as the $X_i$ are generated by correlated sequences of $Y_j^*$’s.

To estimate $\text{Var}(\hat{H}^b(p_k, \vec{y}_i))$ I divide my bootstrap sample of $X_i$’s into $M$ equal segments. Let $\hat{H}^b_m(p_k, \vec{y}_i)$ be the bootstrap estimate constructed using the $m$-th segment of the Bernoulli sequence. Clearly, $\hat{H}^b(p_k, \vec{y}_i)$ is the average over $\hat{H}^b_m(p_k, \vec{y}_i)$’s, and the variance of $\hat{H}^b(p_k, \vec{y}_i)$ can be given by $\frac{1}{M^{\frac{1}{2}}} \Sigma_l$ where $\Sigma$ is the covariance matrix of the $\hat{H}_m$’s. I estimate $\Sigma_m$ by the sample covariance of the segments $X_m$ and $X_l$ of the original Bernoulli sequence.

This method also allows me to estimate the covariance $\text{Cov}(\hat{H}^b(p_k, \vec{y}_i), \hat{H}^b(p_{k-1}, \vec{y}_i))$, which I need to evaluate to form the standard errors for my valuation estimates. Calling the Bernoulli variables generating $\hat{H}^b(p_{k-1}, \vec{y}_i)$ $Z_j$, I proceed in an analogous fashion to calculating the variance, and base my estimates on sample covariance of the segments $X_m$ and $Z_l$.

See Efron and Tibshirani (1998)
In practice, I found that the “jackknife-after-bootstrap” and the delta-method standard errors for the estimated marginal valuations to be similar in magnitude.

C An analytical example

Here is a simple case in which the optimality condition can be used to derive the optimal bidding strategies in the Wilson model with no resale incentives. To the extent of my knowledge, this is the only analytically solved example for the discriminatory price auction game in which bidders have private information and submit continuous bid functions.\(^{49}\)

Suppose there are only 2 bidders, and that their signals are distributed exponentially, with \(F(s) = e^{\lambda s}, s \leq 0\). Also assume that their true demand for T-bills is linear in price and their signal:

\[
D(p, s_i) = \alpha + \beta p + \gamma s_i
\]  

(45)

Let’s first conjecture that their bid schedule will also be linear:

\[
y(p, s_i) = a + bp + cs_i
\]  

(46)

where \(a, b, c\) will be determined as functions of \(\alpha, \beta, \gamma, \lambda\). Taking the linear bid function as given,

\[
H(p, y) = e^{\lambda\left[1 - \frac{y - a - bp}{c}\right]}
\]  

(47)

and

\[
H_p(p, y) = -\frac{\lambda b}{c}e^{\lambda\left[1 - \frac{y - a - bp}{c}\right]}
\]  

(48)

Substituting into the optimality condition:

\[
a + bp + cs_i = \alpha + \beta(p - \frac{c}{\lambda b}) + \gamma s_i
\]  

(49)

Equating coefficients, we get:

\[
y(p, s_i) = \alpha - \frac{\gamma}{\lambda} + \beta p + \gamma s_i
\]  

(50)

Immediately we see that both bidders shade their demands by the amount \(\frac{\gamma}{\lambda}\).

\(^{49}\)Wilson provides some analytic examples for the uniform price auction.
References


