Liquidity Premia in Dynamic Bargaining Markets

By
Pierre-Olivier Weill
Stanford University
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Stanford Institute for Economic Policy Research
Stanford University
Stanford, CA  94305
(650) 725-1874

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Pierre-Olivier Weill

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Abstract

This paper develops a search-theoretic model of the cross-sectional distribution of asset returns, abstracting from risk premia and focusing exclusively on liquidity. In contrast with much of the transaction-cost literature, it is not assumed that different assets carry different exogenously specified trading costs. Instead, different expected returns, due to liquidity, are explained by the equilibrium cross-sectional variation in the distribution of ownership. The qualitative predictions of the model are consistent with much of the empirical evidence. An analysis of the dynamic impact of unexpected news sheds light on time variation in liquidity.

Keywords: Liquidity premia, Search

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1 Introduction

Why do different assets earn different expected returns? One fundamental reason is that they may bear different risks. Many empirical studies, however, suggest that risk characteristics cannot explain all variations in expected returns. After controlling for risk premia, expected returns appear to be positively related to bid-ask spreads, and negatively related to turnover, dollar trading volume, and market capitalization. These patterns suggest that returns are related to liquidity, broadly defined as the ease of buying and selling. Liquidity is reflected in small trading costs, measured for instance by the bid-ask spread, and associated with the opportunity to buy and sell large quantities in a short time, near the quoted price. These properties may be proxied by turnover, trading volume, or market capitalization.

This paper provides a dynamic asset pricing model in which cross-sectional variation in asset returns is exclusively due to liquidity differences. Our main objective is to explain and reproduce some of the qualitative relationships documented by the empirical literature, following a modeling strategy of Duffie, Gârleanu and Pedersen [2001]. As in their model, trade is decentralized: Investors search for each other, meet in pairs, and bargain over prices. In this environment, liquidity is naturally defined as the ease of buying and selling asset. An asset is said to be easier to buy (or to sell) if a seller (or a buyer) is more likely to be found in a short time.

In the present model, as opposed to Duffie, Gârleanu and Pedersen [2001], many different assets are traded. Investors allocate their fixed budgets of search efforts to the various assets. They recognize that the value of searching for a particular asset is related to the likelihood of finding a counterparty for that asset in a short time. The first-order condition of the associated search optimization problem is key to the model’s implications, as it reflects how the likelihood of finding an asset is priced in equilibrium. Namely, in equilibrium, investors are indifferent between searching for alternative traded assets, under natural technical conditions. This indifference property gives rise to a distribution of “liquidity premia.” An asset that is easier to find is sold at a higher price. If all assets pay the same dividend rate, the model predicts that an asset with more shareholders is easier to find, and thus has a lower expected return. Among assets with the same quantity of shareholders, the model predicts that an asset earning a relatively large proportional dividend rate is sought more aggressively. This reduces the fraction of investors that are willing to sell, and makes the asset harder to find. In equilibrium, such an asset has a lower price, other effects held equal.

In traditional Walrasian asset-pricing models with liquidity effects such as those of Amihud and Mendelson [1986], Constantinides [1986], Vayanos [1998], and Huang [2002], assets can be bought and sold instantly, but differ by an exogenously given transaction cost. A more liquid asset is defined as one with a smaller transaction cost. In these models, cross-sectional variation in asset
returns is explained by exogenously specified differences in transaction costs. A main contribution of this paper is to explain cross-sectional variation in asset returns without relying on an exogenously specified cross-sectional variation in transaction costs. Although, in the model proposed here, the search technology is the same for all assets, heterogeneous bid-ask spreads arise endogenously. Cross-sectional variation in asset returns is explained by cross-sectional variation in an asset’s fundamentals: its dividend rate and the quantity of its shareholders.

Our analysis could not be conducted in the one-asset model of Duffie, Gärleanu and Pedersen [2001], which examines the impact of liquidity on asset prices only by comparative-statics results. For instance, in the one-asset model, an increase in the quantity of shareholders results in a positive shift of the supply curve, and thus decreases the price of the asset. In the multiple-assets model, we can keep the total quantity of shareholders constant, and study an equilibrium in which some assets have more shareholders than others. This isolates a liquidity effect: An asset with more shareholders is easier to find, and has a higher price.

Search-theoretic approaches to liquidity have been explored in the monetary literature, following Kiyotaki and Wright [1989]. Most notably, Wallace [2000] focuses on the relative liquidity of intrinsically worthless assets (currency) and assets earning a positive dividend (bonds). The model we present here makes no room for currency, and focuses on assets with relatively homogeneous characteristics. This paper is closely related to the independent work of Vayanos and Wang [2002]. They provide a two-asset extension of Duffie, Gärleanu and Pedersen [2001] with heterogeneous investors and partially segmented markets in order to study liquidity differences between on-the-run and off-the-run bonds. By contrast, the focus here on the cross-sectional distribution of asset returns prompts us to address an arbitrary number of assets. We restrict attention to homogenous investors in order to relate liquidity to asset fundamentals. We allow investors to search simultaneously for several assets, by allocating their search effort to the various assets.

The last section of the paper addresses time variation in liquidity. Specifically, we study the dynamic impact of unexpected news about “asset fundamentals.” Good news about an asset is represented by an unexpected permanent increase in its dividend rate. When the news is announced, investors start aggressively searching for this asset, causing a temporary increase in its trading volume. As the asset is aggressively purchased, it becomes progressively harder to buy. Once this decrease in liquidity compensates for the increase in dividend rate, trading volume goes back to normal, and the economy slowly approaches its new steady-state. Our study suggests that time variation in liquidity has a small impact on the level of prices (analogous to Constantinides [1986]), but may have a non-negligible impact on capital gains, hence on returns.
2 Trading Many Assets

This section presents the basic model, in which investors cannot buy and sell assets instantly. Rather, they allocate search resources to asset-specific “trading specialists,” who search for counterparties. When two investors meet, they bargain over the terms of trade. (The specialists could bargain on their behalves.)

2.1 The Economic Environment

This subsection describes the model setup.

Information, Preferences and Technology

Time is treated continuously, and runs forever. The economy is populated by a continuum of risk-neutral infinitely-lived investors whose total measure is normalized, without loss of generality, to 1. An investor has, at each time, either a high or a low liquidity need, modeled by the level of a randomly varying shock to her marginal utility for consumption. Specifically, with a high liquidity need, an investor values immediate lumps of consumption (“cash”) over streams of consumption (“dividends”) by a factor of $1/(1 - \alpha)$ for some $\alpha \in (0, 1)$. With a low liquidity need, this factor is 1. This creates gains from trade: An investor with a high liquidity need is willing to sell her asset to an investor with a low liquidity need, in exchange for a lump of consumption. Investors switch randomly, and pair-wise independently, from a high liquidity need to a low liquidity need with intensity\(^1\) $\lambda_d$, and from a low need to a high need with intensity $\lambda_h$. Any two investors have pair-wise independent liquidity-need processes. In order to allow for side payments, investors are endowed with a technology to instantly produce lumps of consumption for each other, at unit marginal cost. Lastly, investors can hold at most one unit of any asset, and cannot shortsell.

Assets Fundamental Characteristics

The set of asset types is $\{1, \ldots, K\}$. An asset of type $k$ is an indivisible consol bond that pays the constant dividend rate $d_k$ forever. An investor is permitted to hold at most one unit of an asset. Since an asset is indivisible, $s_k$ is the fraction of the population holding asset $k$. Crucially, this indivisibility assumption induces heterogeneous asset holdings: in equilibrium, different investors hold different assets. (Other segmentation stories might be applied.) In order to make the demand side of the economy nontrivial, we assume that $s = \sum_{k=1}^{K} s_k < 1$.

\(^1\)Specifically, liquidity need is a Markov chain. If an investor has high liquidity need, the distribution of the next switching time to the low-need state is exponential with parameter $\lambda_d$. The successive switching times are independent.
Definition 1 (Assets Fundamental Characteristics.) A distribution of ownership is some $S = (s_1, \ldots, s_K) \in \mathbb{R}_+^K$, quantities of shareholders of each asset, such that $\sum_{k=1}^K s_k = s$. A collection of dividend rates is some $D = (d_1, \ldots, d_K) \in \mathbb{R}_+^K$.

**Investor Types**

An investor’s type is made up of her liquidity need (high $h$, or low $l$), and her ownership status (owner $ok$, or nonowner $n$), for each asset type $k \in \{1, \ldots, K\}$. Hence, the set of investor types is

$$I = \{ln, hn, lo1, \ldots, loK, ho1, \ldots, hoK\}. \quad (1)$$

For each $i \in I$, we let $\mu_i$ denote the fraction of investors of type $i$, and, given the asset fundamentals and the trading environment (to be defined), we let $V_i$ denote the continuation utility of an investor of type $i$. A precise definition of $V_i$ is provided in Appendix 2.

**Random Matching**

At any point in time, each investor is endowed with a unit mass of “trading specialists” who search for specific trading counterparties, in a sense that is now to be described. A trading specialist of type $(i, j) \in I^2$ works for an investor of type $i$, and specializes in contacting specialists working for investors of type $j$. Thus, contacts that could result in a trade occur only between specialists of types $(i, j)$ and $(j, i)$.

An investor of type $i$ maintains on her “trading staff” a quantity $\nu_{ij}$ of specialists of type $(i, j)$, subject to the resource constraint $\sum_{j \in I} \nu_{ij} \leq \bar{\nu}$, which we take to be 1 as a normalization. Thus, the fraction of specialists of type $(i, j)$ in the entire specialist population is $\mu_i \nu_{ij}$. A given specialist makes contacts with other specialists pair-wise independently at Poisson arrival times, with intensity $\lambda > 0$. Contacts are also pair-wise independent with the liquidity-need processes. Given a contact, because of the random matching assumption, the probability that the contact is made with a specialist of type $(i, j)$ is $\mu_i \nu_{ij}$. That is, conditional on making a contact, all trading specialists in the entire specialist population are “equally likely” to be contacted. Adapting the usual random-matching assumption that the Law of Large Numbers applies (see, for instance, Diamond [1982]), contacts between specialists of types $(i, j)$ and $(j, i)$, for $i \neq j$, occur continually at a total (almost sure) rate of

$$\mu_i \nu_{ij} \lambda \mu_j \nu_{ji} + \mu_j \nu_{ji} \lambda \mu_i \nu_{ij} = 2\lambda \mu_i \nu_{ij} \mu_j \nu_{ji}. \quad (2)$$

The first term on the left-hand side of (2) is the total rate of contacts made by all specialists of type $(i, j)$, and received by specialists of type $(j, i)$. Specifically,
each specialist of the mass \( \mu_i \nu_{ij} \) of specialists of type \((i, j)\) makes contacts at rate \( \lambda \), and such contacts are received by some specialist of type \((j, i)\) with probability \( \mu_j \nu_{ji} \). Similarly, the second term is the total rate of contact made by specialists of type \((j, i)\) and received by specialists of type \((i, j)\).

For each investor of type \(i\), \( \lambda_{ij} \equiv \lambda \nu_{ij} \) is the intensity of contacts with some other specialists, made by the mass \( \nu_{ij} \) of specialists of type \((i, j)\). Thus, we can view an investor of type \(i\) as endowed with a budget \( \lambda > 0 \) of search effort and allocating some intensity \( \lambda_{ij} \) to the search for investors of type \(j\), subject to the constraint \( \sum_{j \in I} \lambda_{ij} \leq \lambda \). With this new notation, adopted for the remainder of the paper, the total (almost sure) rate of contact between investors of types \(i\) and \(j\) is

\[
2\mu_i \mu_j \frac{\lambda_{ij} \lambda_{ji}}{\lambda}.
\]  

An investor maintaining trading specialists can be viewed as an investment firm with separate units that trade specific securities. A typical unit trades securities of a specific industry, such as “telecom” or “entertainment,” or trades securities with a specific payoff structure, such as fixed-income or derivatives. Specialization in trading reflects the costs of collecting and processing information regarding the supply and demand of assets, as well as the fundamentals of the underlying cash flows.

This search-theoretic model abstracts from a decentralized security market, such as the NASDAQ or some other over-the-counter markets. One may argue that, in these markets, search frictions are largely overcome by marketmakers who stand ready to buy and sell assets. However, as Duffie, Gárateanu and Pedersen [2001] show, trading frictions remain relevant if trade is bilateral and investors meet marketmakers sequentially. In such environments, search frictions determine the reservation values of investors bargaining with marketmakers, which in turn affect bid and ask prices.

### 2.2 Equilibrium

We now study the decisions of investors: whether or not to trade in a given encounter, and how to allocate search intensity across types of trading encounters. We then describe the dynamics of the distribution of types. Lastly, we define an equilibrium.

**Trade Among Investors**

Trade between investors of types \(i\) and \(j\) occurs at a strictly positive rate if (a) the gain from trade from such a pair is strictly positive,\(^2\) and (b) these two types

\(^2\)An arbitrarily small transaction cost rules out trade when the gain is zero.
of investors maintain trading specialists who are searching for each other, that is, if \( \lambda_i \lambda_j > 0 \).

In equilibrium, we anticipate that the gains from trade are strictly positive between the following types of investor pairings. First, when a high-liquidity-need owner (one of type \( hok \)) contacts a low-liquidity-need non-owner (of type \( ln \)) the \( hok \) investor may sell her asset to the \( ln \) investor, in exchange for a lump of consumption. Second, when an \( hok \) investor contacts an \( loj \) investor, they may swap assets, and the \( loj \) investor may simultaneously transfer a lump of consumption to the \( hok \) investor. These lumps of consumption are instantly produced, at unit marginal cost.

We anticipate an equilibrium in which \( hok \) investors do not maintain trading specialists who search for \( loj \) investors, but only trading specialists who search for \( ln \) investors. In other words, the net utility of searching for an asset swap will turn out to be strictly less than the net utility of searching for an outright sale. Hence, an \( hok \) investor allocates all of her search intensity to the search for \( ln \) investors. On the other hand an \( ln \) investor allocates intensities, denoted \( \lambda_1, \ldots, \lambda_K \), to searches for investors of respective types \( h01, \ldots, h0K \).

**Definition 2** A search intensity allocation is some \( \Lambda \in \mathbb{R}_+^K \) with \( \sum_{k=1}^K \Lambda_k \leq \lambda \).

The terms of trade between an \( ln \) and an \( hok \) investor arise in a simple Nash bargaining game. The total surplus of such a transaction is \( V_{lok} - V_{ln} - (V_{hok} - V_{hn}) \equiv \Delta V_{lk} - \Delta V_{hk} \). We study those equilibria in which the seller receives a fixed fraction \( q < 1 \) of the total surplus.\(^3\) This implies that the price of asset \( k \) is, in an equilibrium,

\[
p_k = q \Delta V_{lk} + (1 - q) \Delta V_{hk}.
\]

**Investors’ Problems and The Distribution of Types**

We first characterize the equilibrium continuation utilities \( V_i, i \in \{1, \ldots, I\} \). We use the unit “lump of consumption” as a numeraire. As shown in Appendix 2, these solve the system of Bellman Equations:

\[
rV_{ln} = \max_{\lambda_1, \ldots, \lambda_K} \left\{ \lambda_u(V_{hn} - V_{ln}) + 2 \sum_{k=1}^K \lambda_k \mu_{hok}(V_{lok} - V_{ln} - p_k) \right\}
\]

\[
rV_{lok} = d_k + \lambda_u(V_{hok} - V_{lok})
\]

\[
rV_{hok} = (1 - \alpha)d_k + \lambda_d(V_{lok} - V_{hok}) + 2 \lambda_k \mu_{ln}(V_{hn} - V_{hok} + p_k)
\]

\[
rV_{hn} = \lambda_d(V_{hn} - V_{hn})
\]

\(^3\)Allowing \( q \) to depend on the pair type would not change our main results. This point is illustrated by the computations of section 4.2.
for all \( k \in \{1, \ldots, K\} \). The maximization in (5) is subject to \( \sum_{k=1}^{K} \lambda_k \leq \lambda \) and \( \lambda_k \geq 0 \), for all \( k \in \{1, \ldots, K\} \). The upper-case and lower-case notation is used to distinguish the search intensity \( \Lambda_k \) that will prevail in equilibrium for all investors of type \( ln \), from the intensity \( \lambda_k \) that is to be chosen by an individual investor of type \( ln \), taking others’ search intensities as given.

Given a search-intensity allocation \( \Lambda \), the distribution \( \mu \equiv (\mu_{ln}, \mu_{hn}, \mu_{hok}, \mu_{lok}) \) of types solves the system

\[
\begin{align*}
0 &= \lambda_d \mu_{hn} - \lambda_u \mu_{hn} - 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok} \\
0 &= \lambda_d \mu_{hok} - \lambda_u \mu_{hok} + 2\Lambda_k \mu_{tn} \mu_{hok} \\
0 &= \lambda_u \mu_{lok} - \lambda_u \mu_{hok} - 2\Lambda_k \mu_{tn} \mu_{hok} \\
0 &= \lambda_u \mu_{tn} - \lambda_u \mu_{hn} + 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok} \\
s_k &= \mu_{hok} + \mu_{lok} \\
1 &= \sum_{k=1}^{K} (\mu_{hok} + \mu_{lok}) + \mu_{tn} + \mu_{hn},
\end{align*}
\]

for \( k \in \{1, \ldots, K\} \). The system (9)-(12) implies that, in a steady-state, for each type, the rate of change of the quantity of investors of that type is zero. For instance, in (9), \( \lambda_d \mu_{hn} \) is the instantaneous flow of investors of type \( hn \) migrating to the \( ln \) type, \( \lambda_u \mu_{hn} \) is the instantaneous flow of investors of type \( ln \) migrating to the \( hn \) type, and \( 2\Lambda_k \mu_{tn} \mu_{hok} \) is the instantaneous flow of investors of type \( ln \) who buy an asset of type \( k \), migrating to the \( lok \) type.

\textit{Steady-State Equilibrium}

\textbf{Definition 3} A steady-state equilibrium is a collection \( V = (V_{ln}, V_{lok}, V_{hok}, V_{hn})_{1 \leq k \leq K} \) of continuation utilities, a distribution \( \mu = (\mu_{ln}, \mu_{lok}, \mu_{hok}, \mu_{hn})_{1 \leq k \leq K} \) of types, and a search intensity allocation \( \Lambda \), such that

(i) Steady-State: Given \( \Lambda \), \( \mu \) solves the system (9)-(14).

(ii) Optimality: Given \( \Lambda \), \( V \) and \( \Lambda \) solve the system (5)-(8) of Bellman equations.

In what follows, we restrict attention to those steady-state equilibria in which all assets are traded, that is, with \( \Lambda_k > 0 \) for all \( k \). Since (5) is a linear program, this amounts to a focus on equilibria in which \( ln \) investors are indifferent between searching for any two assets. The first-order condition of program (5) is
\[ \mu_{hok}(1 - q)(\Delta V_{lk} - \Delta V_{hk}) = \mu_{haj}(1 - q)(\Delta V_{lj} - \Delta V_{hj}), \]

for all \((k, j) \in \{1, \ldots, K\}^2\), meaning that the marginal utility of spending an additional unit of search intensity on a given asset must be equated across assets. This marginal utility is decomposed as follows: Conditional on establishing a contact, a seller of asset \(k\) is found with probability \(\mu_{hok}\). Then, the buyer receives a fraction \(1 - q\) of the transaction surplus \(\Delta V_{lk} - \Delta V_{hk}\).

We may interpret the total transaction surplus as the bid-ask spread, in the following sense. We consider the economy in a steady-state equilibrium and we introduce an “infinitesimal” market-maker.\(^4\) If this marketmaker can make take-it-or-leave-it offers to investors, he charges \(\Delta V_{lk}\) to buyers of asset \(k\) (the ask price), and pays \(\Delta V_{hk}\) to sellers of asset \(k\) (the bid price). Following this interpretation, condition (15) implies that an asset that is easier to find (with a larger \(\mu_{hok}\)) has a narrower bid-ask spread. This suggests a negative relationship between liquidity and bid-ask spread.

\section{Analysis of the Model}

In this section, we provide technical conditions under which an equilibrium exists and is unique. Then, we analyze the pricing implications of the model. Namely, we show how cross-sectional variation in asset prices is explained by cross-sectional variation in asset fundamentals, the dividend rate and the quantity of shareholders.

\subsection{Existence and Uniqueness}

We first analyze the steady-state distribution of types. Second, in order to prove the existence of an equilibrium, we study the indifference conditions (15).

\textit{Steady-State Distribution of Types}

In this paragraph, we study the system (9)-(14), given a search intensity allocation \(\lambda\). Since equation (13) implies that the sum of (10) and (12) is zero, we can eliminate (10). Similarly, since equation (14) implies that the sum of equations (9) to (11) is zero, we can eliminate (12). We finally obtain the reduced system

\(^4\)Considering marketmakers that are not “infinitesimal” would change the equilibrium outcome, as in Duffie, Gärleanu and Pedersen [2001].
\[0 = \lambda_u s_k - (\lambda_u + \lambda_d) \mu_{hok} - 2\Lambda_k \mu_{tn} \mu_{hok}\]  
(16)  
\[0 = \lambda_d (1 - s) - (\lambda_u + \lambda_d) \mu_{tn} - 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok}\]  
(17)  
\[\mu_{lok} = s_k - \mu_{hok}\]  
(18)  
\[\mu_{hn} = 1 - s - \mu_{tn},\]  
(19)  
for \(k \in \{1, \ldots, K\}\). The study of (16)-(19) presented in Appendix 1 shows the following proposition.

**Proposition 1** Given a search intensity allocation \(\Lambda\), the system (16)-(19) has a unique solution \(\mu = (\mu_{hn}, \mu_{lok}, \mu_{hn}, \mu_{hok})_{1 \leq k \leq K} \in [0, 1]^{2K+2}\).

**Indifference Conditions**

The equilibrium conditions can be written as a system of \(K-1\) equations in \(K-1\) unknowns; namely, the indifference conditions (15) may be viewed as an equation \(f(\Lambda) = 0 \in \mathbb{R}^{K-1}\), to be solved for \(\Lambda\). This equation is as follows. Through (16)-(19), we map a search intensity allocation \(\Lambda\) into a unique stationary distribution \(\mu\) of types. Then, solving the linear system (5)-(7) of Bellman equations, we may write \(\mu_{hok} (1 - q)(\Delta V_k - \Delta V_{hk}) = w_k(\Lambda)\), for some function \(w_k(\cdot)\). We express the “search-indifference” marginal condition (15) as

\[w_k(\Lambda) = W,\]  
(20)  
for \(k = 1, \ldots, K\), and for some positive constant \(W\) to be determined. In order to solve for the equilibrium \(W\), we may first solve the system (20) for \(\Lambda\), for any given \(W > 0\). This allows us to write \(\Lambda_k = l_k(W)\), for some functions \(l_k(\cdot)\). The equilibrium \(W\) solves

\[\sum_{k=1}^{K} l_k(W) = \lambda.\]  
(21)  
We use (21) as a “necessary condition” for equilibria. In order to derive (21), we combine the Bellman equations (5)-(8) with equation (16) to show that \(w_k(\Lambda)\) is defined implicitly by

\[rw_k(\Lambda) = \alpha (1-q) \mu_{hok} d_k - w_k(\Lambda) \left(\frac{\lambda_u s_k}{\mu_{hok}} - 2\Lambda_k (1 - q) \mu_{tn}\right) - 2\lambda (1-q) \mu_{hok} W.\]  
(22)  
Then, equation (16) implies that

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\[ 2\Lambda_k \mu_{ln} = \frac{\lambda_u s_k}{\mu_h\alpha} - (\lambda_u + \lambda_d). \]  

(23)

Substituting (23) into (22), imposing \( w_k(\Lambda) = W \), and rearranging gives

\[ \frac{\lambda_u s_k q}{(1 - q)\alpha d_k \mu_{hok}^2} + r + \frac{(1 - q)(\lambda_u + \lambda_d)}{(1 - q)\alpha d_k} \frac{1}{\mu_{hok}} + \frac{2\lambda}{\alpha d_k} = \frac{1}{W}. \]  

(24)

This quadratic equation allow us to write \( \mu_{hok} = m_k(W) \), for some \( W > 2\lambda/\alpha d_k \) and for some continuous and increasing function \( m_k(\cdot) \). Combining (16) and (17), we find that

\[ \mu_{ln} = y - s + \sum_{k=1}^{K} \mu_{hok}. \]  

(25)

We substitute (25) into (23) to find that \( l_k(W) \) of (21) is defined implicitly by

\[ 2l_k(W) \left( y - s + \sum_{k=1}^{K} m_k(W) \right) = \left( \frac{\lambda_u s_k}{m_k(W)} - (\lambda_u + \lambda_d) \right), \]  

(26)

which, in turn, shows that \( \sum_{k=1}^{K} l_k(W) = \lambda \) if and only if

\[ 2\lambda \left( y - s + \sum_{k=1}^{K} m_k(W) \right) - \sum_{k=1}^{K} \left( \frac{\lambda_u s_k}{m_k(W)} - (\lambda_u + \lambda_d) \right) = 0. \]  

(27)

The left-hand side of (27) is increasing in \( W \) because \( m_k(\cdot) \) is increasing for each \( k \). Hence, (27) uniquely characterizes a candidate equilibrium.

**Proposition 2 (Uniqueness.)** There is at most one equilibrium with \( \Lambda \gg 0 \).

We first analyze the case of identical asset characteristics. We fix a distribution \( \tilde{S} = (\hat{s}/K, \ldots, \hat{s}/K) \), of ownership, and a collection \( \tilde{D} = (\hat{d}, \ldots, \hat{d}) \), of dividends, for some positive constants \( \hat{s} \) and \( \hat{d} \). We show the existence of a symmetric equilibrium with \( \hat{\Lambda}_k = \lambda/K \), following Duffie, Gârleanu and Pedersen [2001]. Then, in order to prove local existence, we apply the Implicit Function Theorem to equation (27), around this symmetric equilibrium.

**Proposition 3 (Existence.)** Let \( \tilde{S} = (\hat{s}/K, \ldots, \hat{s}/K) \), and \( \tilde{D} = (\hat{d}, \ldots, \hat{d}) \). Then, there is a neighborhood \( N \subset \mathbb{R}^{2K} \) of \( (\tilde{S}, \tilde{D}) \), such that, for all \( (S, D) \in N \), there is an equilibrium, and all such equilibria have \( \Lambda \gg 0 \).

*Proof.* If the assets have identical characteristics, it is natural to guess that there is a symmetric equilibrium, with \( \mu_{hok} = \mu_{ho}/K \) and \( \hat{\Lambda}_k = \lambda/K \). The equilibrium equations are those of Duffie, Gârleanu and Pedersen [2001], with “\( \lambda \)” there being replaced here by “\( \lambda/K \)” Their results imply that investors’ values are strictly positive, and that

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there are strictly positive gains from trade between investors of types \( ln \) and \( hok \). Furthermore, since assets have identical characteristics, there is no gain from swapping assets. Thus, \( hok \) investors strictly prefer searching for a sell with an \( ln \) investor to searching for a swap with an \( laj \) investor, for all \( j \in \{1, \ldots, K\} \). Since the left-hand side of (27) is strictly increasing in \( W \), we can apply the Implicit Function Theorem. This provides a neighborhood \( N \subset \mathbb{R}^{2K} \) of \((S, D)\), such that, for all \((S, D) \in N\), there exists a candidate equilibrium \( W = h(S, D) \), for some continuous function \( h(\cdot, \cdot) \). The other candidate equilibrium objects \((V, \mu, \Lambda)\) are easily expressed as continuous functions of \( W \) and thus as continuous functions of \((S, D)\). The indifference conditions (20) are satisfied by construction. All other relevant inequalities hold by continuity. In Appendix 3, we show how to choose the neighborhood \( N \) such that all equilibria in \( N \) have \( \Lambda \gg 0 \).

The proof shows in particular that, if assets characteristics are sufficiently homogeneous, \( hok \) investors are not searching for swaps. This follows from the fact that the net utility of swapping two assets with nearly identical characteristics is close to zero and turns out to be strictly less than the net utility of searching for an outright sale.

The intuition for the local absence of equilibria with non-traded assets is as follows. If asset \( k \) is not traded, then the fraction of sellers of this asset is \( s_k\lambda_u/(\lambda_u + \lambda_d) \). On the other hand, the fraction of sellers of a traded asset \( j \neq k \) is \( \mu_{haj} < s_j\lambda_u/(\lambda_u + \lambda_d) \). If \( k \) and \( j \) have sufficiently similar characteristics, an investor strictly prefers to search for asset \( k \) because it is easier to find. This contradicts the assumption that \( k \) is not traded.

Does there always exist an equilibrium in which all assets are traded? We provide a partial answer, in a two-asset economy. Specifically, we show that if the assets have sufficiently different supplies, there cannot be an equilibrium in which both are traded, in the following sense.

**Proposition 4 (Non-Existence.)** Consider a two-asset economy \((K = 2)\). Assume that these assets pay the same dividend rate, that Asset 1 is in supply \( s_1 > 0 \), and that Asset 2 is in supply \( s - s_1 > 0 \). Then there is a \( \varepsilon > 0 \) such that, for any \( s_1 < \varepsilon \), an equilibrium with \( \Lambda \gg 0 \) cannot exist.

Existence in Proposition 3, and non-existence in Proposition 4, are proved by studying how equation (27) depends on \((S, D)\). When asset characteristics are sufficiently similar, we can show that the equation has a solution. Alternatively, when the quantity of shareholders of an asset is sufficiently small relative to quantities of shareholders of other assets, we can show that there is no solution.

An equilibrium may fail to exist because, when \( s_1 \) is small, the probability of finding a seller is even smaller. An investor is willing to search for this asset only if she is compensated by a sufficiently low price. If \( s_1 \) is small enough, the appropriate compensation results in a negative price, and thus cannot be the basis of an equilibrium.
3.2 The Cross-Section of Asset Returns

The objective of this subsection is to show how the cross-sectional variation in asset returns is explained by cross-sectional variation in the asset fundamentals, the quantities of shareholders, and the dividend rates.

Here, the cross-sectional variation in asset returns is not explained by an exogenously specified cross-sectional variation in transaction costs, in contrast with the Walrasian models of Amihud and Mendelson [1986], Constantinides [1986], Vayanos [1998], and Huang [2002]. In our model, because of the search friction, investors cannot find buyers and sellers of specific assets instantly, and because investors are impatient, the likelihoods of finding those buyers and sellers in a short time are reflected in prices. We view the cross-sectional variation in the likelihood of finding buyers and sellers as the natural counterpart of a cross-sectional variation in transaction costs. This cross-sectional variation is not, however, exogenously specified. Rather it arises endogenously and is explained by the assets’ characteristics.

In order to derive the cross-sectional relationship between returns and asset characteristics \((S, D)\), we use the following three equations. The main equation is a version of the price-setting equation \((4)\). Using the Bellman equations for \(V_n\) and \(V_{lok}\), we write \((4)\) as

\[
p_k = \frac{d_k}{r} - \frac{2\lambda W}{r} - \left(1 + \frac{\lambda_u}{(1-q)r}\right) \frac{W}{\mu_{hok}}.
\]

(28)

The right-hand side is increasing in \(\mu_{hok}\). In other words, an asset that is easier to find (one with larger \(\mu_{hok}\)) is sold at a higher price. The second key equation \((24)\) is of the form

\[
A \frac{s_k}{d_k \mu_{hok}^2} + B \frac{1}{d_k \mu_{hok}} + C \frac{1}{d_k} = \frac{1}{W},
\]

(29)

for some positive constants \(A\), \(B\), and \(C\), which do not depend on \(k\). The third key equation is easily derived from \((16)\), and relates \(\Lambda_k\) to the distribution of types and \(s_k\). That is,

\[
\frac{\mu_{hok}}{s_k} = \frac{\lambda_u}{\lambda_u + \lambda_d + 2\Lambda_k \mu_{ln}}.
\]

(30)

The quantity \(\Lambda_k \mu_{ln}\) has several interpretations. First, it represents the demand side of the market. The larger is \(\Lambda_k\), the more search occurs for asset \(k\), and the easier it is to sell this asset. It is natural to ask whether an asset that is easier to sell is also easier to find. That is, can one view \(\Lambda_k \mu_{ln}\) as an increasing function of \(\mu_{hok}\)? Equation \((30)\) shows that the answer depends on the quantity \(s_k\) of shareholders, and is thus indeterminate at this stage of the analysis. Second, \(\Lambda_k \mu_{ln}\) is related to the mean holding period of asset \(k\). An investor holds asset \(k\) until she switches to a state of high liquidity need, which takes an average time of
\( \lambda_{u}^{-1} \). Then, she searches for a buyer for an expected time of \((2 \Lambda_{k} \mu_{ln})^{-1} \). Thus, an asset with larger \( \Lambda_{k} \) has shorter average holding periods.

Equipped with (28), (29), and (30), the following paragraphs relate liquidity premia to exogenous asset characteristics.

**Size Effects**

In this paragraph, we assume that all assets pay the same dividend rate and differ only in the quantities of their shareholders. As the following analysis makes clear, there is a sense in which cross-sectional variation in the quantity of shareholders captures cross-sectional variation in market capitalization, \( p_{k} s_{k} \) for asset \( k \). In particular, this model provides some theoretical foundations for the “size effects” that have been documented in the empirical literature (see Banz [1981]). When \( D = (d, \ldots, d) \), equation (29) takes the form

\[
F_{1}(s_{k}, \mu_{hok}) = \frac{1}{W},
\]

for some function \( F_{1}(\cdot, \cdot) \) that is increasing in \( s_{k} \) and decreasing in \( \mu_{hok} \). This implies that \( \mu_{hok} \) is increasing in \( s_{k} \). In other words, an asset with more shareholders is easier to find, is sold at a higher price, and has a lower gross return \( R_{k} = 1 + d_{k}/p \). In order to derive a relationship between the quantity \( s_{k} \) of shareholders and the mean holding period \( \lambda_{u}^{-1} + (2 \Lambda_{k} \mu_{ln})^{-1} \), we write equation (29) as

\[
G_{1}\left(s_{k}, \frac{\mu_{hok}}{s_{k}}\right) = \frac{1}{W},
\]

for some function \( G_{1}(\cdot, \cdot) \) that is decreasing in \( s_{k} \) and decreasing in \( \mu_{hok}/s_{k} \). This implies that \( \mu_{hok}/s_{k} \) is a decreasing function of \( s_{k} \). From (30), it follows that \( \Lambda_{k} \mu_{ln} \) is an increasing function of \( s_{k} \). In other words, an asset with more shareholders has a shorter mean holding period. Lastly, since the total rate of contact between buyers and sellers of asset \( k \) is \( 2 \Lambda_{k} \mu_{ln} \mu_{hok} \), an asset with more shareholders also has a larger trading volume. The above discussion is summarized in

**Proposition 5 (Size Effects.)** Assume that \( D = (d, \ldots, d) \). In equilibrium, \( s_{k} > s_{j} \) implies that \( \mu_{hok} > \mu_{hoj} \), \( \Lambda_{k} > \Lambda_{j} \), \( p_{k} > p_{j} \), \( p_{k} s_{k} > p_{j} s_{j} \), \( R_{k} < R_{j} \), and \( \Delta V_{ik} - \Delta V_{hk} < \Delta V_{ij} - \Delta V_{hj} \).

In words, an asset with more shareholders is easier to find, easier to sell, has a shorter mean holding period, has a larger trading volume, has a higher price, has a larger market capitalization, has lower return, and has a narrower bid-ask spread. The model thus generates a “size effect,” that is, a negative relationship between returns and market capitalization. An asset with a larger quantity of shareholders has a higher price, thus a larger market capitalization, and a lower return.
This model also generates a positive relationship between returns and holding periods with ex-ante identical investors, because returns and holding periods are both negatively related to a common exogenous “liquidity” factor, the quantity of shareholders. By contrast, in Amihud and Mendelson [1986], the holding period itself is an exogenous parameter. A positive relationship between returns and holding periods also arises endogenously in general equilibrium models with transaction costs, such as those of Vayanos and Villa [1999] or Huang [2002], but for a different reason. In these models, assets can be bought and sold instantly, and an investor choose to hold assets with larger transaction costs for a longer periods. These assets, in equilibrium, have higher expected returns. In our model, an asset cannot be bought and sold instantly, and an asset with a higher return is harder to sell, and thus has a longer mean holding period.

**Narrow Ownership**

In this paragraph, we assume that \( s_k d_k \), the total amount of dividends distributed per unit of time for asset \( k \), does not depend on \( k \). This promotes one way to think about asset ownership: A firm with a smaller \( s_k \) and a larger \( d_k \) is more “narrowly held.”

Part of the gain from trade reflects the opportunity cost of not completing the trade between the current pair of \( l_h \) and \( h_k \) investors. Specifically, if there is no trade, then during the following inter-contact-time period the asset continues to be held by the \( h_k \) investor, who has lower marginal utility for the dividend rate \( d_k \), resulting in an “instantaneous opportunity cost” of \( \alpha d_k \). The larger is the dividend rate, the larger is the opportunity cost of the foregone trade, and the larger is the gain from trade. Thus, because it has a larger dividend rate, a more narrowly held asset gives, once found, a larger surplus to the buyer. Equilibrium imposes that investors, who receive a fixed share of the transaction surplus, are indifferent between searching for all traded assets. This implies that liquidity adjusts to compensate for differences in dividend rates. Namely, a more narrowly held asset, which has a larger dividend rate, must be harder to find than a less narrowly held asset.

We now formalize this intuition. First, since \( s_k d_k \) does not depend on \( k \), equation (29) can be written

\[
F_2(s_k, \mu_{hok}) = \frac{1}{W},
\]

for some function \( F_2(\cdot, \cdot) \) that is increasing in \( s_k \) and decreasing in \( \mu_{hok} \). This implies that a more narrowly held asset is harder to find, and is sold at a lower price relative to its “fundamental value” \( d_k/r \). The effect of narrow ownership on the mean holding period is found by rewriting (29) as

\[
G_2\left(s_k, \frac{\mu_{hok}}{s_k}\right) = \frac{1}{W},
\]
for some function $G_2(\cdot, \cdot)$ that is increasing in $s_k$ and decreasing in $\mu_{ho}/s_k$. This implies, together with (30), that a narrowly held asset has a shorter mean holding period.

**Proposition 6 (Narrow Ownership.)** Assume that $s_k d_k$ does not depend on $k$. Then $s_k < s_j$ and $d_k > d_j$ imply that $\mu_{ho} < \mu_{hoj}$, $\Lambda_k > \Lambda_j$, $p_k - d_k/r < p_j - d_j/r$, and $\Delta V_k - \Delta V_{hk} > \Delta V_{ij} - \Delta V_{hj}$.

In words, a narrowly-held asset is harder to find but easier to sell. It is more aggressively sought, and its mean holding period is shorter. It has a lower price relative to its fundamental value, and its bid-ask spread is larger. The effect on returns is ambiguous at this level of analysis. Numerical work, not reported here, suggest that a more narrowly held asset has a lower return, at least for the range of parameters considered.

## 4 Cross-Sectional Returns and Liquidity

The objective of this section is to confront the qualitative predictions of the theoretical model with empirical evidence. After a brief review of the empirical literature that relates cross-sectional asset returns to liquidity factors, we compute an equilibrium of the theoretical model. For a “random” cross section of 200 assets, we show how returns are related to liquidity factors.

### 4.1 Empirical Evidence

Amihud and Mendelson [1986] and Amihud and Mendelson [1989] propose an empirical analysis of the “liquidity-premium” hypothesis. They study monthly returns on portfolios of NYSE stocks, over the period 1961-1981. They proxy for liquidity with the relative bid-ask spread, in line with their theoretical model, in which the relative bid-ask spread is an exogenous characteristic of the asset. Controlling for risk premia using a CAPM beta (Sharpe [1964]), and for market-capitalization, they show that there is a significant positive relationship between relative bid-ask spreads and expected returns. Subsequent studies have criticized aspects of their methodology, raising two main methodological questions. The first question is how to proxy for liquidity. Petersen and Fialkowski [1994] argued that the bid-ask spread is a poor measure of trading costs. They study market orders for 144 stocks listed on the NYSE, over the three-months period November 1990 to January 1991. They show that 50% of transactions do not occur at the quoted bid-ask spread.\(^5\) The second question has been the degree of control

\(^5\)This observation is consistent with the model of Duffie, Garleanu and Pedersen [2001], in which transactions between investors and a monopolistic marketmaker occur at the quoted bid-ask spread, and transactions between pairs of investors occur within the spread.
for other factors than liquidity, most notably for systematic risk. When controlling for risk premia using CAPM betas, the econometrician is testing jointly the liquidity-premium hypothesis and the CAPM theory. A significant measured liquidity effect may reflect a failure of the CAPM and not necessarily evidence of a liquidity premium.

*Alternative Measures of Liquidity*

Eleswarapu [1997] studies monthly excess returns on portfolios of NASDAQ stocks, over the period 1973-1990. He controls for risk premia, using a CAPM beta, and for market capitalization. Contrary to the evidence from the earlier cited NYSE study, most trades on NASDAQ occur at the quoted bid-ask spread. Furthermore, the observed variation in bid-ask spread is much larger across NASDAQ stocks than across NYSE stocks. This suggests that a test of NASDAQ data may have more power to reject the null of no liquidity premium. Eleswarapu’s results indicate that liquidity is indeed priced.

Brennan and Subrahmanyam [1996] study monthly excess returns on portfolios of NYSE stocks, over the period 1984-1991. They do not rely on the quoted bid-ask spread, but estimate fixed and proportional trading costs from intraday transactions. In their cross-sectional regressions, the estimated trading cost is positively related to return, after controlling for risk using the three factors of Fama and French [1993].

Other authors such as Haugen and Baker [1996], Hu [1997], and Brennan, Chordia and Subrahmanyam [1998], relate liquidity to trading activity. An asset is said to be more liquid if it is traded more frequently and in larger (dollar) quantities. This may reflect the opportunity to conduct a large trade without a large price impact.

Haugen and Baker [1996] study monthly returns on individual stocks listed in the Russell 3000 index, over the period 1979-1993. In their regressions, they use four liquidity factors: market capitalization, market price per share, a ratio of monthly dollar trading volume to market capitalization, as well as the trend of this ratio. In addition, they include factors indicating risk, price level, and growth potential, as well as sector variables and technical factors. The ratio of monthly dollar trading volume to market capitalization, which one may interpret as a measure of turnover, appears to be the most important liquidity factor and to be negatively related to expected returns.

Hu [1997] studies monthly returns of stocks traded on the Tokyo Stock Exchange, over the period 1976-1993. He measures liquidity with trading turnover, the number of shares traded divided by the number of shares outstanding. His regressors include market capitalization, book-to-market ratio, and cash-flow-to-price ratio. He finds a statistically and economically significant negative relationship between expected returns and turnover.

Brennan, Chordia and Subrahmanyam [1998] study monthly excess returns
on individual stocks traded on NYSE and NASDAQ, over the period 1966-1995. They control for liquidity with the dollar trading volume, for risk with either the three factor of Fama and French, or the method of Connor and Korajczyk [1988] that is based on asymptotic principal components. The non-risk factors used by Brennan, Chordia and Subrahmanyam [1998] include the price, a measure of dividend yield, and lagged returns. They find that the dollar trading volume of an asset is negatively related to its expected excess return.

Controlling for Risk

These cited studies use various factor models in order to control for risk premia. As mentioned above, a significant measured liquidity effect may reflect a misspecification of the factor model. We now review some work that attempts to avoid this criticism.

Amihud and Mendelson [1991] compare expected returns among securities with similar risk characteristics. They focus on two government securities, treasury bills (shorter maturity), and treasury notes (longer maturity). They match bills and notes with the same maturity date, cash-flow, and risk. Their sample covers 37 trading days, between April and November 1987, and includes bills and notes with less than 6 months to maturity. Amihud and Mendelson’s presumption is that notes are much less liquid than bills of the same time to maturity: Since notes have been traded for a longer period, part of their supply has been “locked away” in investors portfolios. Amihud and Mendelson find that notes have significantly higher yields to maturity and larger bid-ask spreads, supporting their presumption that notes are less liquid than bills. Crabbe and Turner [1995] apply this methodology to corporate bonds and medium-term notes issued by four large borrowers, over the 1987-1992 period. They find no significant liquidity effect.

Kadlec and McConnell [1994] study the prices of NASDAQ securities that obtained a NYSE listing during the 1980-1989 period. Since trading costs appear to be smaller on the NYSE than on the NASDAQ (see, among others, Huang and Stoll [1996]), the authors expect after-listing prices to reflect better market liquidity. They measure liquidity with the bid-ask spread, controlling for the increase in shareholders base (Merton [1987]) and for the “good news” associated with a NYSE listing. They find a positive relationship between bid-ask spreads and returns.

4.2 Cross-Sectional Returns: An Example

This section presents a numerical example suggesting that the predictions of the theoretical model developed in this paper are qualitatively consistent with much of the evidence from the empirical literature.

An equilibrium of the model is computed for a randomly generated economy
of $K = 200$ asset types.\textsuperscript{6} The asset characteristics, $s_k$ and $d_k$, are drawn independently from uniform distributions on the intervals $[s_1, s_2]$ and $[d_1, d_2]$, respectively. The bargaining powers $q_1, \ldots, q_K$, of sellers of assets $1, \ldots, K$, respectively, are also drawn independently from an uniform distribution on the interval $[q_1, q_2]$. This is a simple way to check the robustness of the results to the introduction of unobserved asset heterogeneity. The equilibrium return $d_k/p_k$ is plotted against various measures of liquidity used in the empirical literature. The measures have direct counterparts in our theoretical model. The relative bid-ask spread is $1 - \Delta V_{hk}/\Delta V_{lk}$. The dollar trading volume is $p_k \mu_h \Lambda_k \mu_{lb}$. The turnover is $\mu_h \Lambda_k \mu_{lb}/s_k$. The market capitalization (size) is $p_k s_k$. The values of the exogenous parameters are as in Table 1.

The unit of time is one year. Assuming that the stock market opens 250 days a year and that there are 10 trading hours per day, $\lambda = 2500$ means that an investor establishes a contact once per hour, on average. The discount rate $r$ is 5%. Given the chosen uniform distribution for $s_k$, the expected aggregate supply of assets, $E \left( \sum_{k=1}^{K} s_k \right)$, is 0.1. As in Duffie, Gârleanu and Pedersen [2001], an investor has a high liquidity need, on average, for 1 year out of every 11 years.

Figure 1 displays the results of the computations. Returns and bid-ask spreads are positively related. In contrast with the theoretical results of Amihud and Mendelson [1986], the relationship is almost linear and not concave. Consistently with the empirical evidence, returns are negatively related to market capitalization, turnover, and trading volume. The holding period is positively related to returns. There is a slightly negative relationship between returns and dividends rates.

\textsuperscript{6}We solve (27) for $W$. This can be done quickly since equation (24), characterizing $m_k(W)$, is quadratic. Once $W$ is found, the remaining equilibrium objects are easily computed using the various equations derived during the existence proof.
5 The Impact of Surprises on Liquidity

We now turn from the steady-state cross-sectional distribution of asset returns, toward some of the time-series implications of our model of liquidity, using some examples that shed light on the dynamic impact of unexpected news on prices and returns. News regarding “fundamentals,” such as the dividend rates, cause investors to deviate from their steady-state search allocations. This has an impact on the distribution of investors’ types, and as a consequence the likelihood of finding buyers and sellers of given types. Specifically, when unexpected news regarding fundamentals is announced, investors are no longer indifferent between searching for any two assets, and the distribution of investors’ types must adjust in order to eventually restore indifference. Because investors need to contact each other in order to trade, the distribution of investors’ types cannot adjust instantly.

We construct a numerical approximation of the search-intensity allocation
Table 2: Parameter Values used in Studying the Dynamic Impact of Unexpected News.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contact Intensity</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Intensity of Switch to High</td>
<td>$\lambda_u$</td>
</tr>
<tr>
<td>Intensity of Switch to Low</td>
<td>$\lambda_d$</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>$r$</td>
</tr>
<tr>
<td>Liquidity Shock</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>Number of Assets</td>
<td>$K$</td>
</tr>
<tr>
<td>Measure of Shareholders</td>
<td>$s_k$</td>
</tr>
<tr>
<td>Dividend Rate</td>
<td>$d_k$</td>
</tr>
<tr>
<td>Bargaining Power</td>
<td>$q_k$</td>
</tr>
</tbody>
</table>

along the equilibrium path. Because an investor's search-intensity allocation solves a linear program, it typically features jumps from corner to corner. In order to ensure the smoothness of the policy function, and to apply standard solution methods, we incorporate a small, strictly concave penalty function into the search optimization problem of an $ln$ investor. This provides an approximation of this optimization problem. We conjecture that the equilibrium constructed on the basis of this approximation approximates an equilibrium for the actual underlying model. In any case, there are in practice costs to changing the allocation of search efforts. In Appendix 4, we describe the numerical method in detail.

We consider a two-asset economy ($K = 2$). Our two experiments share the following features. At any $t < 0$, the economy is assumed to be at the steady-state equilibrium associated with the parameter values of Table 2. At $t = 0$, an unexpected piece of news regarding the assets’ characteristics is announced. The news is unexpected in the sense that it was not allowed for by investors at $t < 0$.\(^7\) Figures\(^8\) 2 to 5 display the “transitional dynamics” of equilibrium quantities, for $t > 0$.

**Unexpected Permanent Increases in Dividend Rates**

\(^7\)One may consider an alternative setup in which “fully rational” investors allow for a one time Poisson-arrival of a piece of news. This would increases the computational complexity of the exercise since the characteristic of the economy at $t < 0$ (the “initial conditions”) would depend on the path of the economy at $t > 0$ (the “transitional dynamics.”) In order to compute the initial conditions, one would need to solve a fixed-point problem on top of the current procedure which solves for transitional dynamics.

\(^8\)The unit of time of the rates and intensities in Table 2 is one year. In our numerical experiments, the economy is close to its new steady state after a few hours of trading. As a result, the time unit in the Figures is in “hours.” Years are converted in hours assuming 250 trading days per year and 10 hours of trading per day.
This experiment describes the effect of an unexpected 10% permanent increase in the dividend rate $d_1$, which one could interpret as an unexpected piece of good news regarding the long run profitability of the firm issuing Asset 1. The results are displayed in Figures 2 and 3.

![Graphs of Sellers of Asset 1, Intensity allocated to Asset 1, Sellers of Asset 2, Intensity allocated to Asset 2, Turnovers of Asset 1 and 2, and Time (Hours)](image)

Figure 2: The Effects of a Permanent Increase in $d_1$.

The unexpected dividend increase temporarily makes Asset 1 more attractive. As a result, investors search for Asset 1 with an intensity close to their full budget $\lambda$. This causes $\mu_{ho1}$ to decrease rapidly and $\mu_{ho2}$ to increase rapidly. In other words, Asset 2 becomes relatively more liquid than Asset 1. This change in relative liquidity compensates for the change in dividend so as to make investors nearly indifferent between searching for both assets. Near indifference is achieved after about an hour of heavy trading, by which time investors’ search intensity allocations have moved close to their new steady-state values.

In the new steady-state, $\mu_{ho1} < \mu_{ho2}$, but $\Lambda_1 > \Lambda_2$. Because investors seek it more aggressively, Asset 1 is harder to find and easier to sell. Liquidity deteriorates for buyers but improves for seller. The “net” effect may be measured by turnover: Asset 1 is more heavily traded than Asset 2 and this is reflected in returns.
Figure 3: The Effects of a Permanent Increase in $d_1$.

At $t = 0$, the prices of both assets jump close to the levels that are their new steady-state values. Because Asset 1 is initially more liquid than in the new steady-state, its price is slightly larger than its steady-state value. It subsequently decreases, as liquidity deteriorates. The price impact of time variation in liquidity appears to be small, in line with the results of Constantinides [1986]. Since, on the other hand, the time derivatives of prices are not small, the impact on instantaneous returns (through capital gains), is not negligible in the short run.

**Unexpected Temporary Increases in Dividend Rates**

The previous paragraph considers a permanent dividend increase, that one could view as a piece of news regarding the long-run relative profitability of the firm issuing Asset 1. We now consider a temporary dividend rate increase. Specifically, the dividend $d_1$ is increased at $t = 0$ by 10% from its old level, and then decays exponentially, with a one-hour half life. (That is, $d_1 = \nu d_1$, where $\nu$ is set so that $d_1(0^+) - d_1(1) = 1/2 (d_1(0^+) - d_1(0^-))$.) Results are displayed in Figures 4 and 5.

As in the previous experiment, Asset 1 is more attractive and initially investors
search for it more aggressively. Its liquidity deteriorates while the liquidity of Asset 2 improves. As Asset 1 becomes harder to find, and has a smaller dividend, Asset 2 becomes the most attractive. Then, investors seek Asset 2 more aggressively. The liquidity of Asset 1 improves, while that of Asset 2 deteriorates. The humped-shaped pattern for the time path of $\mu_{h02}(t)$ is expected because the initial increase in the dividend $d_1$ rate is temporary, and all variables eventually revert toward their steady-state values. The reversion in liquidity is also illustrated by the turnover: The turnover of Asset 1 is initially larger than that of Asset 2, but becomes smaller after about half an hour of heavy trading.

6 Conclusion

This paper uses a search-theoretic model to study the impact of heterogeneity in asset liquidity on the cross section and the time series of asset returns. Although the search technology is the same for all assets, heterogeneous bid-ask spreads arise endogenously. Cross-sectional variation in returns is explained by cross-
sectional variation in share ownership. Numerical results suggest that some key qualitative properties of the model are consistent with empirical evidence.

The out-of-steady-state dynamics shed light on the short-term impact of unexpected news. The price impact is generally small, but the return impact is not negligible. The model suggests that even moderate unexpected news may create temporary but sharp increases in trading volume.

Further work might use a search-and-bargaining model to conduct a quantitative study of the cross section of asset returns. It would be helpful to extend the current framework in order to incorporate both risk premia and stochastic variation in aggregate liquidity. Such an analysis may provide theoretical foundations for factor models that are commonly used in empirical studies.
References


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Appendix 1: Dynamics of the Type Distribution

In this appendix, we study the dynamics of the distribution of types. We first solve for the steady-state, and then proves its local stability. For a given search intensity allocation $\Lambda$, the distribution $\mu(t) = (\mu_{tn}(t), \mu_{tok}(t), \mu_{hok}(t), \mu_{hn}(t))_{1 \leq k \leq K}$ of types solve

\[
\begin{align*}
\dot{\mu}_{tn} &= \lambda_d \mu_{tn} - \lambda_u \mu_{tn} - 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok} \\
\dot{\mu}_{tok} &= \lambda_d \mu_{tok} - \lambda_u \mu_{tok} + 2 \Lambda_k \mu_{tn} \mu_{hok} \\
\dot{\mu}_{hn} &= \lambda_u \mu_{hn} - \lambda_d \mu_{hn} + 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok} \\
\dot{\mu}_{hok} &= \lambda_u \mu_{hok} - \lambda_d \mu_{hok} - 2 \Lambda_k \mu_{tn} \mu_{hok} \\
s_k &= \mu_{hok} + \mu_{tok} \\
1 &= \sum_{k=1}^{K} (\mu_{hok} + \mu_{tok}) + \mu_{tn} + \mu_{hn},
\end{align*}
\]

where $\dot{\mu} = d\mu(t)/dt$, and time arguments are suppressed. Since equation (39) implies that the sum of (36) and (38) is zero, we can eliminate (36), the ODE for $\mu_{tok}$. Similarly, since equation (40) implies that the sum of equations (35) to (38) is zero, we can eliminate (37), the ODE for $\mu_{hn}$. We obtain the equivalent system

\[
\begin{align*}
\dot{\mu}_{hok} &= \lambda_u s - (\lambda_u + \lambda_d) \mu_{hok} - 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok} \\
\dot{\mu}_{tn} &= \lambda_d (1 - s) - (\lambda_u + \lambda_d) \mu_{tn} - 2 \sum_{k=1}^{K} \Lambda_k \mu_{tn} \mu_{hok} \\
\mu_{tok} &= s_k - \mu_{hok} \\
\mu_{hn} &= 1 - s - \mu_{tn},
\end{align*}
\]

for $k \in \{1, \ldots, K\}$.

**Steady-State Distribution of Types, Proposition 1**

A steady-state solves equations (41)-(44). Summing equations (41) over $k$, adding equation (42), and imposing the steady-state condition $\dot{\mu} = 0$, we find

\[
\mu_{tn} = \mu_{t0} + y - s,
\]

where $\mu_{t0} \equiv \sum_{k=1}^{K} \mu_{hok}$, and $y \equiv \lambda_d / (\lambda_u + \lambda_d)$. We replace this last equation in (41) to obtain
\[ \mu_{ho} = \frac{\lambda_u s_k}{\lambda_u + \lambda_d + 2\Lambda_k (\mu_{ho} + y - s)}, \quad (46) \]

summing equations (46) over \( k \), we obtain the one equation in one unknown problem

\[ \mu_{ho} - \sum_{k=1}^{K} \frac{\lambda_u s_k}{\lambda_u + \lambda_d + 2\Lambda_k (\mu_{ho} + y - s)} = 0. \quad (47) \]

The left-hand side of this equation is increasing in \( \mu_{ho} \), is negative at \( \mu_{ho} = 0 \), and is positive for \( \mu_{ho} \) large enough; thus, it has a unique solution. Once the solution \( \mu_{ho} \) is found, \( \mu_{hok} \) is uniquely determined by (46), \( \mu_{ln} \) by (45), and finally \( \mu_{lok} \) and \( \mu_{hn} \) by (43) and (44). This procedure characterizes a unique candidate steady-state. Since the steady-state fractions sum to one by construction, we only need to show that they are positive. We proceed as follows. The left-hand side of (47) is positive when evaluated at \( \mu_{ho} = s \) and \( 1 - y \); it is negative when evaluated at \( s - y \). Since the left hand side of (47) is increasing, this shows that

\[ s - y < \mu_{ho} < \min\{s, 1 - y\}. \quad (48) \]

Next, \( s - y < \mu_{ho} \) implies that \( \mu_{ln} > 0 \) and that \( \mu_{hok} < s_k \). Finally, \( \mu_{ho} < 1 - y \) implies that \( \mu_{ln} < 1 - s \) and that \( 0 < \mu_{hn} < 1 \).

**Local Stability**

We now establish that, given \( \Lambda \), the steady-state distribution of types is a locally stable point of the following ODE

\[ \dot{\mu}_{hok} = \lambda_u s_k - (\lambda_u + \lambda_d)\mu_{hok} - 2\lambda\Lambda_k \mu_{ln}\mu_{hok}, \quad (49) \]

\[ \dot{\mu}_{ln} = \lambda_d (1 - s) - (\lambda_u + \lambda_d)\mu_{ln} - 2 \sum_{k=1}^{K} \Lambda_k \mu_{ln}\mu_{hok}, \quad (50) \]

for all \( k \in \{1, \ldots, K\} \). Stacking variables as \( (\mu_{h01}, \ldots, \mu_{hok}, \mu_{ln})' \), the Jacobian of the ODE at the steady-state is

\[ J = - (\lambda_u + \lambda_d)I_{K+1} - D, \quad (51) \]

where \( D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \), \( D_{11} = \text{diag}(2\Lambda_k \mu_{ln}) \), \( D_{12} = [2\Lambda_1 \mu_{h01} \ldots 2\Lambda_K \mu_{h0K}]' \), \( D_{21} = [2\Lambda_1 \mu_{ln} \ldots 2\Lambda_K \mu_{ln}] \), and \( D_{22} = \sum_{k=1}^{K} 2\Lambda_k \mu_{hok} \).

**Lemma 1 (Local Stability.)** The eigenvalues of \( J \) have strictly negative real parts.
Proof. Letting $e$ denote the vector $(1, \ldots, 1)'$, we have $e^\prime D_{11} = D_{21}$ and $e^\prime D_{12} = D_{22}$. We let $x \neq 0$ be an eigenvector of $J$ associated with the eigenvalue $\nu \in \mathbb{C}$. We have

\begin{align}
(\lambda_u + \lambda_d)x_1 + D_{11}x_1 + D_{12}x_2 &= -\nu x_1 \quad (52) \\
(\lambda_u + \lambda_d)x_2 + D_{12}x_1 + D_{22}x_2 &= -\nu x_2. \quad (53)
\end{align}

We multiply equation (52) by $e^\prime$, and subtract equation (53) to obtain

$$
(\lambda_u + \lambda_d + \nu)(e^\prime x_1 - x_2) = 0.
$$

(54)

We distinguish three cases.

Case 1: $e^\prime x_1 \neq x_2$. From (54), it must be that $\nu = -(\lambda_u + \lambda_d) < 0$.

Case 2: $e^\prime x_1 = x_2 = 0$. Then (52) simplifies to $(\lambda_u + \lambda_d + D_{11})x_1 = -\nu x_1$. Thus, it must be that $\nu = -(\lambda_u + \lambda_d + 2\Lambda_k \mu \mu_n) < 0$, for some $k \in \{1, \ldots, K\}$.

Case 3: $e^\prime x_1 = x_2 \neq 0$. Without loss of generality, we assume that $x_2 = 1$. We use (52) to solve, explicitly for $x_{1k}$,

$$
x_{1k} = -\frac{2\Lambda_k \mu \mu_{ok}}{\nu + \lambda_u + \lambda_d + 2\Lambda_k \mu \mu_n}.
$$

(55)

Since the $x_{1k}$ sum to one, it must be that $\text{Re} \left( \sum_{k=1}^K x_{1k} \right) = 1$. Thus, there is one $k \in \{1, \ldots, K\}$ such that $\text{Re}(x_{1k}) > 0$, which is equivalent to $\text{Re}(1/x_{1k}) > 0$ and, from equation (55), to $\text{Re}(\nu) < -(\lambda_u + \lambda_d + 2\Lambda_k \mu \mu_n) < 0$. □

## Appendix 2: Formulating and Solving the Investor’s Problem

This Appendix defines the stochastic control problem faced by an individual investor in a candidate steady-state equilibrium, and verifies that the Bellman equations (5)-(8) are sufficient for optimality.

### A Candidate Steady-State Equilibrium

We first describe a candidate steady-state equilibrium as follows. There is a fraction $\mu_i$ of investors of type $i \in I$, with search intensity allocations $(\Lambda_{ij})_{j \in I}$, consuming at the constant rate $c(i)$.

The liquidity need process of an investor of type $i$ switches with intensity $\lambda^s(i) > 0$. To simplify notations, we describe switches of the liquidity need process as encounters with a “fictitious” investor $s \notin I$. We let $H = I \cup \{s\}$ be the set of possible encounters.
The terms of trade are described by a “transition function” \( \sigma : I \times H \to I \), and a “payment function” \( p : I \times H \to \mathbb{R}_+ \). Specifically, when an investor of type \( i \) meets an investor of type \( j \in H \), they either accept or reject the trade. If both accept, the investor of type \( i \) evolves to type \( \sigma(i, j) \) and makes a payment of \( p(i, j) \) to the investor of type \( j \). If either of them reject the trade, no payment occurs and the investors stay at their preceding respective types. The candidate equilibrium strategy is to always accept the trade. Lastly, when an investor of type \( i \) meets the investor \( s \), he evolves to one of type \( \sigma(i, s) \) and makes the payment \( p(i, s) = 0 \).

**The Investor’s Problem**

We consider an individual investor with initial type \( i_0 \). We fix a measurable space \((\Omega, \mathcal{F})\) and a measurable counting process \( N_t = (N_t(j))_{j \in H} \) where \( N_t(j) \in \mathbb{N} \) counts the numbers of encounters with investors of type \( j \in H \) in the time interval \([0, t]\). The process \( N_t \) is associated with a sequence of encounter times \( 0 = T_0 < T_1 < T_2 < \cdots \) and a sequence of \( H \)-valued random variables \( j_1, j_2, \ldots, j_n, \ldots \) such that, at \( t = T_n \), the investor encounters an investor of type \( j_n \). We let \( \{\mathcal{F}_t^N, t \geq 0\} \) be the internal history of (filtration generated by) the process \( N_t \).

**Definition 4 (Admissible Controls.)** An admissible control is some \((\mathcal{F}_t^N)\)-adapted process \( u_t \equiv (\beta_t^u, \lambda_t^u, t \geq 0) \). The process \( \{\beta_t^u, t \geq 0\} \) is the trading strategy. It is \( \{0, 1\} \)-valued and equal to 1 for all \( t \geq 0 \) such that \( dN_t(s) = 1 \). The process \( \{\lambda_t^u, t \geq 0\} \) is the search intensity strategy. It is \( \mathbb{R}_+^I \)-valued, left-continuous with right limit (LCRL), and such that \( \sum_{j \in I} \lambda_t^u(j) \leq \lambda \). We let \( \mathcal{U} \) be the set of admissible controls.

The trading strategy describes the decision of accepting \((\beta_t^u = 1)\) or rejecting \((\beta_t^u = 0)\) the prescribed trade in an encounter occurring at time \( t \) (the type of the encounter is known from the information filtration). The search intensity strategy describes how an investor uses her search effort over time. An admissible control \( u \) generates a \( I \)-valued type process \( X_t^u \) as follows:

\[
X_0^u = i_0
\]
\[
X_{T_n}^u, T_n \leq t < T_{n+1}
\]
\[
X_{T_{n+1}}^u = \beta_{T_{n+1}}^u \sigma(X_{T_{n+1}}^u, j_{n+1}) + (1 - \beta_{T_{n+1}}^u) X_{T_{n+1}}^u.
\]

An admissible control \( u \in \mathcal{U} \) is associated with a probability \( P_u \) on \((\Omega, \mathcal{F})\) such that \( N_t \) admits the \((P_u - \mathcal{F}_t^N)\) intensity

\[
\eta_t^u(j) = \lambda_t^u(j) \lambda_j X_j^u \frac{2\mu_j}{\lambda},
\]

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for \( j \in I \), and
\[
\eta_t^u(s) = \lambda^J(X_t^u). \tag{60}
\]

The consumption process associated with the admissible control \( u \) is defined by the stochastic differential equation
\[
dC_t^u = c(X_t^u) \, dt - \sum_{j \in H} p(X_{t^-}^u, j) \beta_t^u(j) \, dN_t(j). \tag{61}
\]

**Definition 5 (Investor's Problem.)** The lifetime utility of an investor with initial type \( i_0 \) applying some admissible control \( u \in \mathcal{U} \) is
\[
\bar{V}_{i_0}(u) = E_0^u \left( \int_0^{+\infty} e^{-r t} \, dC_t^u \right), \tag{62}
\]
The investor's problem is to attain the optimal lifetime utility
\[
V_{i_0} = \sup_{u \in \mathcal{U}} \bar{V}_{i_0}(u). \tag{63}
\]

**Dynamic Programming**

An admissible feedback \( v \) is some \((\delta^v, \theta^v)_{i \in I}\), where \( \delta^v : I \times H \to \{0, 1\} \) and \( \theta^v : I^2 \to \mathbb{R}_+ \), such that, for all \( i \in I \), \( \delta^v(i, s) = 1 \) and \( \sum_{j \in I} \theta^v(i, j) \leq \lambda \). For each \((i, j) \in I \times H\), we let \( \eta^v(i, j) = 2\mu_j \theta^v(i, j) \Lambda_{ji} / \lambda \), and \( \eta^v(i, s) = \lambda^J(i) \). The set of admissible feedbacks is denoted \( \mathcal{V} \). With this notation, the system (5)-(8) of Bellman equations is
\[
rJ(i) = \max_{v \in \mathcal{V}} \left\{ c(i) + \sum_{j \in H} \eta^v(i, j) \delta^v(i, j) \left( J (\sigma(i, j)) - J(i) - p(i, j) \right) \right\}, \tag{64}
\]
for all \( i \in I \). In the text, we solve the Bellman equations (64) and we show that the maximum is achieved for some \( v^* \). This feedback is associated with the admissible control \( u^* \) defined as follows. The trading strategy \( \beta_t^{u^*} \) is defined recursively by
\[
\begin{align*}
\beta_0^{u^*} &= 0, \quad X_0^{u^*} = i_0 \tag{65} \\
\beta_t^{u^*} &= \beta_{T_n}^{u^*}, \quad X_t^{u^*} = X_{T_n}, \quad T_n \leq t < T_{n+1} \tag{66} \\
\beta_{T_{n+1}}^{u^*} &= \delta^v(X_{T_{n+1}}, j_{n+1}) \tag{67} \\
X_{T_{n+1}}^{u^*} &= \beta_{T_{n+1}}^{u^*} \sigma(X_{T_{n+1}}^{u^*}, j_{n+1}) + (1 - \beta_{T_{n+1}}^{u^*}) X_{T_{n+1}}^{u^*}. \tag{68}
\end{align*}
\]
And the search intensity strategy \( \lambda_t^{u^*} \) is
\[
\lambda_t^{u^*} = (\theta^v(X_{t^-}, j))_{j \in I}. \tag{69}
\]
Proposition 7 (Sufficiency of the Bellman Equations.) The supremum utility $V_{i_0}$ is bounded above by $J(i_0)$, and this upper bound is achieved by the admissible control $u^*$. 

Proof. We adapt the proof of Theorem VII, T1 in Brémaud [1981]. We consider an admissible control $u \in \mathcal{U}$ and write

$$J(X_t^u)e^{-rt} = J(X_0^u) + \sum_{0 < T_n \leq t} \left( J(X_{T_n}^u)e^{-rT_n} - J(X_{T_{n-1}}^u)e^{-rT_{n-1}} \right) + J(X_{\tau_t}^u)(e^{-rt} - e^{-r\tau_t}),$$

where $\tau_t = \sup\{T_n, n \geq 0 : T_n \leq t\}$. Equation (70) can be manipulated as follows:

$$J(X_t^u)e^{-rt} = J(X_0^u) + \sum_{0 < T_n \leq t} J(X_{T_{n-1}}^u) + J(X_{\tau_t}^u)(e^{-rt} - e^{-r\tau_t})$$

$$+ \int_0^t \frac{d}{ds}(J(X_s^u)e^{-rs}) ds$$

$$+ \int_0^t \sum_{j \in H} \beta_t^u(j)(J(\sigma(X_s^u, j)) - J(X_s^u)) e^{-rs} dN_s(j)$$

$$= J(X_0^u) + \int_0^t \left( -rJ(X_s^u) + \sum_{j \in H} \beta_t^u(j)\eta_t^u(j)(J(\sigma(X_s^u, j)) - J(X_s^u)) \right) e^{-rs} ds$$

$$+ \int_0^t \sum_{j \in H} \beta_t^u(j)(J(\sigma(X_s^u, j)) - J(X_s^u)) e^{-rs}(dN_s(j) - \eta_t^u(j)ds).$$

Adding $\int_0^t e^{-rs} dC_s^u$ to both sides, we obtain

$$\int_0^t e^{-rs} dC_s^u + J(X_t^u)e^{-rt} =$$

$$J(X_0^u) + \int_0^t (-rJ(X_s^u) + c(X_s^u)) e^{-rs} ds$$

$$+ \int_0^t \left( \sum_{j \in H} \beta_t^u(j)\eta_t^u(j)(J(\sigma(X_s^u, j)) - J(X_s^u) - p(X_s^u, j)) \right) e^{-rs} ds$$

$$+ \sum_{j \in H} \int_0^t \beta_t^u(j)(J(\sigma(X_s^u, j)) - J(X_s^u) - p(X_s^u, j)) e^{-rs}(dN_s(j) - \eta_t^u(j)ds).$$

(71)

Since $\beta_t^u(j)(J(\sigma(X_s^u, j)) - J(X_s^u) - p(X_s^u, j)) e^{-rt}$ is a bounded $\mathcal{F}_t^N$-predictable process, it follows by theorem II, T8 in Brémaud [1981] that the last term on the right-hand
side of (71) is a martingale. Taking expectations on both sides, and using the Bellman equation (64), we find
\[
E_0^u \left( \int_0^t e^{-rs} dC_s^u + J(X_t^u) e^{-rt} \right) \leq J(X_0^u),
\]
with equality for \( u = u^* \), the admissible control associated with the \( v^* \) that solves (64). Letting \( t \) go to infinity proves that \( \tilde{V}_k(u) \leq J(X_0^u) \), with equality if \( u = u^* \). ■

**Appendix 3: Proof of Proposition 3 and 4**

This Appendix discusses equilibrium existence without imposing \( \Lambda \gg 0 \), proving in particular the local uniqueness of an equilibrium in which all assets are searched. The system of Bellman equation

\[
\begin{align*}
   rV_{ln} & = \lambda_u (V_{hn} - V_{ln}) + 2\lambda W \\
   rV_{lok} & = d_k + \lambda_u (V_{lok} - V_{lok}) \\
   rV_{hn} & = \lambda_d (V_{ln} - V_{hn}) \\
   rV_{hok} & = (1 - \alpha)d_k + \lambda_d (V_{lok} - V_{hok}) + 2\Lambda_k \mu_{hok} (1 - q_k) (\Delta V_{lk} - \Delta V_{hk}),
\end{align*}
\]

for all \( k \in \{1, \ldots, K\} \). The net utility \( W \) of searching for an asset, and the search intensity allocation \( \Lambda \) solve

\[
W = \max_{\sum \lambda_k = \lambda} \frac{1}{\lambda} \sum_{k=1}^K \lambda_k \mu_{hok} (\Delta V_{lk} - \Delta V_{hk}),
\]

The steady-state distribution of types solves

\[
\begin{align*}
   0 & = \lambda_u s_k - (\lambda_u + \lambda_d) \mu_{hok} - 2\Lambda_k \mu_{ln} \mu_{hok} \\
   0 & = \lambda_d (1 - s) - (\lambda_u + \lambda_d) \mu_{ln} - 2 \sum_{k=1}^K \Lambda_k \mu_{ln} \mu_{hok} \\
   \mu_{lok} & = s_k - \mu_{hok} \\
   \mu_{hn} & = 1 - s - \mu_{ln},
\end{align*}
\]

for \( k \in \{1, \ldots, K\} \). We let \( \sigma_k \equiv \Delta V_{lk} - \Delta V_{hk} \) be the bid-ask spread. Combining equations (74) to (77), we obtain

\[
r\sigma_k = \alpha d_k - (\lambda_u + \lambda_d) \sigma_k - 2\Lambda_k \mu_{ln} q_k \sigma_k - 2\lambda W,\]

for \( k \in \{1, \ldots, K\} \). We let \( w_k \equiv \mu_{hok} (1 - q_k) \sigma_k \) be the net utility of searching for asset \( k \). We multiply (83) by \( (1 - q_k) \mu_{hok} \), and (79) by \( (1 - q_k) \sigma_k \). We subtract the two resulting equations to obtain

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\begin{align}
rw_k &= \alpha d_k (1 - q_k) \mu_{hok} - w_k \left( q_k \frac{\lambda_u s_k}{\mu_{hok}} + (\lambda_u + \lambda_d) (1 - q_k) \right) + 2 \lambda (1 - q_k) \mu_{hok} W, \\
\end{align}

which can be rearranged as

\begin{align}
\frac{1}{W} &= \frac{\lambda_u s_k q_k}{(1 - q_k) \alpha d_k \mu_{hok}^2 W} + \frac{w_k}{(1 - q_k) \alpha d_k \mu_{hok}} + \frac{r}{(1 - q_k) \alpha d_k \mu_{hok}} + \frac{2 \lambda}{\alpha d_k}, \quad (85)
\end{align}

Equipped with equation (85), we show the following Lemma.

**Lemma 2** A collection \((V, \mu, \Lambda, W)\) is a solution of (74)-(82) if and only if it solves (74)-(77), (79)-(82), and

\begin{align}
\frac{1}{W} = \min \left\{ \frac{\lambda_u s_k q_k}{(1 - q_k) \alpha d_k \mu_{hok}^2 W} + \frac{w_k}{(1 - q_k) \alpha d_k \mu_{hok}} + \frac{r}{(1 - q_k) \alpha d_k \mu_{hok}} + \frac{2 \lambda}{\alpha d_k} \right\}. \quad (86)
\end{align}

*Proof.* That (78) implies (86) is obvious from the above derivation. In order to prove the converse, we observe that (85) has been derived without using equation (78). It is sufficient to show that (85) and (86) imply equation (78). Subtracting (86) to (85) shows that

\[
\frac{w_k}{W} - 1 \leq 0,
\]

with equality for the \(k\) realizing the minimum in (86), which is (78). \(\blacksquare\)

In other words, keeping the other equilibrium equations unchanged, we can replace (78) by (86). We also make use of the following result:

**Lemma 3**

\[
\sum_{k=1}^{K} \Lambda_k \Rightarrow 2 \lambda (y - s + \mu_{ho}) - \sum_{k=1}^{K} \left( \frac{\lambda_u s_k}{\mu_{hok}} - (\lambda_u + \lambda_d) \right) = 0. \quad (87)
\]

*Proof.* This follows from simple manipulation of (79), and from the observation, made in Appendix 1, that \(\mu_n > 0\). \(\blacksquare\)

We use both Lemma 2 and Lemma 3 to show the following.

**Proof of the second part of Proposition 2.** We consider the collection \((\hat{S}, \hat{D})\) of asset characteristics, and a strict subset \(T\) of \(\{1, \ldots, K\}\). We construct a candidate equilibrium in which only assets \(k \in T\) are traded. All traded assets realize the minimum in (86). Through this equation, we map a candidate \(W\) into a unique collection

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of $\mu_{ho} = m_k(W) < \lambda_u/(\lambda_u + \lambda_d)\hat{s}/K$, for some $W > 2\lambda/\sigma_d$, for some strictly increasing function $m_k(\cdot)$ and for all $k \in T$. For all $k \notin T$, $\Lambda_k = 0$ and (79) imply that $\mu_{ho} = \lambda_u/(\lambda_u + \lambda_d)\hat{s}/K$. We write equation (87) as

$$2\left(y - \hat{s} + \sum_{k \in T} m_k(W) + \sum_{k \notin T} \frac{\lambda_u \hat{s}_k}{\lambda_u + \lambda_d} - \sum_{k \in T} \left(\frac{\lambda_u \hat{s}_k}{m_k(W)} - (\lambda_u + \lambda_d)\right)\right) = 0. \quad (88)$$

We observe that the left-hand side of (88) is strictly increasing in $W$, goes to $-\infty$ as $W$ goes to zero, and is equal to $2\lambda y(1 - \hat{s}) > 0$ for the $W$ that solves $m_k(W) = \lambda_u \hat{s}/K/(\lambda_u + \lambda_d)$. Thus, equation (88) has a unique solution. One easily verifies that the unique candidate equilibrium cannot be an equilibrium: Since a non-traded asset is easier to find than a traded asset, and since all assets have identical fundamental characteristics, (86) is (strictly) violated. Since (88) is strictly increasing in $W$, we can apply the Implicit Function Theorem around $(\hat{S}, \hat{D})$. There is a neighborhood $\hat{N} \subset \mathbb{R}^{2T}$ of $(\hat{S}, \hat{D})$, such that, for all $(S, D) \in \hat{N}$, equation (88) has a unique solution. Furthermore, the corresponding $(V, \mu, \Lambda, W)$ are continuous with respect to $(S, D)$. Lastly, since (86) is violated at $(\hat{S}, \hat{D})$, it is also violated locally, by continuity. This argument provides a neighborhood $N_T \subset \mathbb{R}^{2T}$ of $(\hat{S}, \hat{D})$, within which an equilibrium in which only $k \in T$ are traded cannot exist. We repeat the same argument for all strict subsets $T$ in the collection $T$ of non-empty strict subsets of $\{1, \ldots, K\}$. Finally, we let $\hat{N} = \cap_{T \in T} N_T$, be the intersection of the resulting collection of neighborhoods, which is a neighborhood of $(\hat{S}, \hat{D})$ since $T$ is finite. By construction, for all $(S, D)$ in this neighborhood, there is no equilibrium in which only a strict subset of assets are traded. $\blacksquare$

**Proof of Proposition 4** We consider a two-asset economy with $s_1 = \varepsilon > 0$ and $s_2 = s - \varepsilon > 0$. We show that, if $\varepsilon$ is small enough, we cannot construct a candidate equilibrium in which $\Lambda \gg 0$. If such an equilibrium exists, both assets realize the minimum in (86). Considering the indifference condition for Asset 1, we observe that since $\mu_{ho1} \leq s_1 = \varepsilon$, the candidate $W$ goes to zero as $\varepsilon$ goes to zero. From the indifference condition for Asset 2, this implies that $\mu_{ho2}$ goes to zero as $\varepsilon$ goes to zero. But then equation (87) cannot hold. $\blacksquare$

**Appendix 4: Numerical Method.**

This Appendix describes the numerical method used to study the impact of unexpected pieces of news. In what follows, we fix the assets’ fundamental characteristics $(S, D)$, and we consider a steady-state equilibrium in which all assets are searched. We let $(V^*, \mu^*, \Lambda^*, W^*)$ be, respectively, the equilibrium steady-state continuation utilities, distribution of types, search intensity allocation, and net utility of searching for assets.

**Dynamic of the Distribution of Type**
The calculations presented at the beginning of Appendix 1 show that the distribution \( \mu \) of types is a solution of the system of ODEs:

\[
\begin{align*}
\dot{\mu}_{hok} &= \lambda_u s_k - (\lambda_u + \lambda_d)\mu_{hok} - 2\lambda_k \mu_{hn}\mu_{hok} \\
\dot{\mu}_{hn} &= \lambda_d (1 - s) - (\lambda_u + \lambda_d)\mu_{hn} - 2 \sum_{k=1}^{K} \lambda_k \mu_{hn}\mu_{hok} \\
\mu_{tOk} &= s_k - \mu_{hok} \\
\mu_{hn} &= 1 - s - \mu_{hn},
\end{align*}
\]

for some initial conditions \( (\mu_{hok}(0), \mu_{tOk}(0), \mu_{hn}(0), \mu_{hn}(0))_{1 \leq k \leq 0} \) in \([0, 1]^{2K+2}\) such that, for all \( k \), \( \mu_{hok} + \mu_{tOk} = s_k \) and \( \mu_{hn} + \mu_{hn} + \sum_{k=1}^{K} \mu_{hok} + \mu_{tOk} = 1 \). In what follows, we let \( \mu_t = (\mu_{hok}(t), \mu_{hn}(t))_{1 \leq k \leq K} \) be the (reduced) distribution of types. The other fractions \( \mu_{tOk} \) and \( \mu_{hn} \) are given by, respectively, equations (91) and (92).

**Penalized Bellman Equations**

Because the search intensity allocation solves a linear program, it typically features jumps from corner to corner. A standard linear approximation relying on the differentiability of investors’ policy function (see Judd [1999]) cannot be used. To ensure smoothness of the policy function, we incorporate a barrier function into the objective of \( bn \) investors. Specifically, we write the system of Bellman equations:

\[
\begin{align*}
\dot{r}V_{hn} &= \lambda_u (V_{hn} - V_{hn}) + 2\lambda W + \dot{V}_{hn} \\
\dot{r}V_{tOk} &= d_k + \lambda_u (V_{hok} - V_{tOk}) + \dot{V}_{tOk} \\
\dot{r}V_{hn} &= \lambda_d (V_{hn} - V_{hn}) + \dot{V}_{hn} \\
\dot{r}V_{hok} &= (1 - \alpha) d_k + \lambda_d (V_{tOk} - V_{hok}) \\
&\quad + 2\lambda_k \mu_{hn} q(\Delta V_{lk} - \Delta V_{hk}) + \dot{V}_{hok},
\end{align*}
\]

where the value \( V_i \) of each type \( i \) and the value \( W \) of searching for assets are implicitly a function of time \( (t) \), and \( \dot{V}_i \) denote the derivative of \( V_i(t) \) with respect to time. The value \( W \) of searching for assets is

\[
W = \max_{\lambda_1, \ldots, \lambda_K} \frac{1}{\lambda} \left( \sum_{k=1}^{K} \lambda_k W_k + \varepsilon W^* \lambda^* \log \left( \frac{\lambda_k}{\lambda^*} \right) \right), \tag{97}
\]

where \( \varepsilon \) is a strictly positive constant and \( W_k \equiv \mu_{hok}(1 - q)(\Delta V_{lk} - \Delta V_{hk}) > 0 \) denotes the net utility of searching for asset \( k \). The maximization in (97) is
subject to $\sum_{k=1}^K \lambda_k \leq \lambda$ and $\lambda_k \geq 0$ for all $k \in \{1, \ldots, K\}$. Our “penalization” of the linear program in (97) amounts to assuming that it is costly to deviate from the steady-state search intensity allocation. The chosen specification (a “barrier function”) guarantees that a solution is unique and interior.

With the penalized Bellman equations, a steady-state is defined, as before, as a collection $(V, \mu, \Lambda) \in \mathbb{R}^{K+4}$ solving equations (89)-(92) and (93)-(97), in which the time derivatives $\dot{V}_i$ and $\dot{\mu}_i$ are set to zero. Clearly, the equilibrium $(V^*, \mu^*, \Lambda^*)$ of the economy without penalization ($\varepsilon = 0$) is a steady state of the economy with penalization ($\varepsilon > 0$).

**Approximating the Search Intensity Allocation**

Simple computations show that the unique solution of (97) is

$$
\lambda_k (\nu(\varepsilon), \varepsilon) = \frac{\varepsilon W^* \Lambda^*_k}{\nu(\varepsilon) - W_k},
$$

(98)

$$
\sum_{k=1}^K \lambda_k (\nu(\varepsilon), \varepsilon) = \lambda
$$

(99)

$$
\nu(\varepsilon) > \max_{1 \leq k \leq K} \{W_k\},
$$

(100)

where $\nu(\varepsilon)$ is the Lagrange multiplier of the constraint $\sum_{k=1}^K \lambda_k \leq \lambda$, and the dependence of the solution $\lambda_k$ on $\nu(\varepsilon)$ and $\varepsilon$ is made explicit in the notation.

**Lemma 4** Let $M = \arg \max_{1 \leq j \leq K} \{W_j\}$. If $k \notin M$, then, as $\varepsilon \to 0$,

$$
\lambda_k (\nu(\varepsilon), \varepsilon) \to 0.
$$

(101)

If, on the other hand, $k \in M$, then, as $\varepsilon \to 0$,

$$
\lambda_k (\nu(\varepsilon), \varepsilon) \to \lambda \frac{\Lambda^*_k}{\sum_{j \in M} \Lambda^*_j}.
$$

(102)

**Proof**. If $k \notin M$, then, from (100), $\nu(\varepsilon) - W_j$ is bounded away from zero. This clearly implies (101). If, on the other hand, $k \in M$, then, letting $W_M = \max_{1 \leq j \leq M} \{W_j\}$, we can write (99) as

$$
\sum_{k \in M} \frac{\varepsilon W^* \Lambda^*_k}{\nu(\varepsilon) - W_M} + h(\varepsilon) = \lambda,
$$

(103)

where $h(\varepsilon)$ goes to zero as $\varepsilon$ goes to zero. This implies (102).■

In words, when $\varepsilon > 0$ is small, the solution of (98)-(100) approximates a solution of the the (linear) program (97) with $\varepsilon = 0$. This property does not guarantee, however, that the equilibrium constructed on the basis of this approximation approximates an equilibrium in which an investor solves the (linear)
program (97) with $\varepsilon = 0$. We conjecture as much, however, and proceed.

**Perfect Foresight Dynamics**

The “natural state” of the dynamic system is made up of the dividend rates $d_t = (d_1(t), \ldots, d_K(t))$, the (reduced) distribution $\mu_t = (\mu_{t_{0:k}}(t), \mu_{tn}(t))_{1 \leq k \leq K}$ of types, and the net utility of searching for each asset $w_t = (W_1(t), \ldots, W_K(t))$. The dividend rates $d_t \in \mathbb{R}^K$ and the distribution of types $\mu_t \in \mathbb{R}^{K+1}$ are the predetermined variables and the net utilities of searching for assets $w_t \in \mathbb{R}^K$ are the non-predetermined variables. We let $y_t \equiv (d_t', \mu_t', w_t') \in (3K + 1) \times 1$. Equations (98)-(100) describe an investor’s search intensity allocation $\Lambda$ in term of a smooth function $L(\cdot)$, with $\Lambda = L(W)$. Hence, the dynamics of the state are described by the system

\[
\begin{align*}
\dot{d}_t &= G \cdot (d_t - d^*) \quad \text{(104)} \\
\dot{\mu}_t &= H(\mu_t, L(w_t)) \quad \text{(105)} \\
\dot{w}_t &= R(d_t, \mu_t, w_t, L(w_t)). \quad \text{(106)}
\end{align*}
\]

The dynamic of the dividend rate (104) is assumed to be autonomous and linear for convenience. Equation (105) represents the ODE (89)-(90) for the distribution of types, and equation (106) follows from simple manipulation of the system (93)-(96 of penalized Bellman equations.

**Linearized Dynamics**

We first check the local uniqueness of the perfect-foresight dynamics by linearizing the system (104)-(106) in a neighborhood of its steady-state. The linearized dynamics are

\[
\begin{bmatrix}
\dot{d}_t \\
\dot{\mu}_t \\
\dot{w}_t
\end{bmatrix} = \begin{bmatrix}
G & 0 & 0 \\
0 & J^H_d & J^L \mu \\
J^R_d & J^R_\mu & J^R \mu + J^R_J w
\end{bmatrix} \begin{bmatrix}
d_t - d^* \\
\mu_t - \mu^* \\
w_t - W^*
\end{bmatrix}, \quad (107)
\]

where $J^f_d$ denotes the Jacobian of some function $f : \mathbb{R}^M \to \mathbb{R}^N$. In order to check the local determinacy of the perfect-foresight equilibrium, we use the eigenvalue decomposition that is standard in linear rational expectations models. (see Buitter [1984] for the continuous-time version.) In all of our numerical examples, we find that $J$ has as many eigenvalues with positive real part as non-predetermined variables, ensuring local determinacy.

The linearization also provides an approximation of the perfect-foresight equilibrium path. We instead propose computations based on a reverse-shooting method. As the figures make clear, reverse shooting with a barrier function provides a smooth approximation of a “bang-bang” policy function. A linearization,
on the other hand, could not capture this “bang-bang” feature.

**Computing Perfect Foresight Equilibrium**

We follow the reverse-shooting method described in Judd [1999]. The perfect-foresight equilibrium path solves the system (104)-(106) of ordinary differential equations, denoted $\dot{y} = g(y)$. We let $y^*$ be the steady-state and $y_0 = \left[ d_0' \quad \mu_0' \quad w_0' \right]'$ be the initial condition. We fix a time horizon $T$. Given a candidate terminal value $y_T$, we solve the ODE $\dot{y} = g(y)$ backward, using a fourth-order Runge-Kutta method. This computation delivers candidate-initial conditions $\tilde{d}_0(y_T)$, $\tilde{\mu}_0(y_T)$, and $\tilde{w}_0(y_T)$. The reverse-shooting method solves the problem

$$\min_{\tilde{d}_0, \tilde{\mu}_0} ||\tilde{d}_0(y_T) - d_0||^2 + ||\tilde{\mu}_0(y_T) - \mu_0||^2,$$

subject to $||y_T - y^*|| < \eta$, where $\eta$ is a small positive number.

In order to solve the program (108), we use a continuation method. Namely, we solve successive programs along a decreasing path $\varepsilon_1 > \varepsilon_2 \cdots > \varepsilon_N = \varepsilon$. The $n$-th version of the optimization program (108) is used as the initial condition of the $n + 1$-st program.

For the study of an unexpected permanent increase of the dividend rate, we set $\varepsilon = 0.001$. For the study of a temporary increase, we set $\varepsilon = 0.003$. 

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