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Optimal Second Price Auctions with Positively Correlated Private Values and Limited Information

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Abstract

Closely related to the work of Athey and Haile (2002), we consider the problem of maximizing revenue in a second-price private-values auction with only limited data from previous auctions. We find that, when bidder values for the object being sold are not assumed to be distributed independently, the optimal levels of the reserve price and (when available) entry fee are not uniquely identified, and can vary over a significant range. Further, under two common representations of positively correlated values, the optimal values of these parameters tend to be lower than analysis assuming independence would have suggested.

While this is written as a theory paper, the well-publicized use of auctions by the government (spectrum auctions, procurement auctions, and others) means that any work on how to best run an auction, and on the data requirements and difficulties in running an auction optimally, has clear policy implications.

1 Introduction

In its primer for prospective sellers, eBay's website defines a reserve price as "the lowest price at which you are willing to sell your item."[1] The obvious purpose of a reserve price is to keep the seller out of sales he would regret. However, a reserve price can be set above the seller's value for the unsold object in an attempt to increase expected revenue. By setting the reserve price above his own value for the good, the seller risks precluding a profitable sale; however, he will sometimes earn a higher price when he does sell, which may offset this risk. The magnitude of the problem can, depending on the auction, be millions of dollars. In New Zealand, the government auctioned off seven television signal licenses in simultaneous second-price sealed-bid auctions without a reserve price. (See Milgrom (forthcoming).) The auctions raised only NZ $2.3 million, despite winning bids exceeding four times that amount; the use of an appropriate reserve price could have increased revenue to nearly $7 million.

In some auctions, the seller can also charge an entry fee; this may allow him to extract more of the buyers' surplus and increase expected revenue.

In a private values auction where the joint probability distribution of bidders' values is known, optimal auction parameters can be calculated as the solution to a straightforward maximization problem. When the entire joint distribution of bidder values is not observed, the problem is no longer straightforward. We ask the question of what bounds can be placed on the optimal parameters of second-price auctions when bidder values are not independent and the full joint distribution of values is unknown. We assume that only the distribution of the second-highest value is known, as from a data set including transaction prices but no other bids.

On the whole, we find that when values are distributed symmetrically but not independently, optimal auction parameters can vary over a wide range. Under two cases of positively correlated values, optimal parameters are still not uniquely identified, but tend to be lower than they would be if values were independent; in particular, we find upper bounds on the optimal reserve price and entry fees, and show that no informative lower bound exists. This suggests that analysis assuming independent values will tend to overstate optimal parameters when values are in fact positively correlated.

2 Related Literature

Athey and Haile (2002, henceforth AH) separate the problem of theoretical identification of a model from the problem of estimation, and define identification this way. Let $\mathbf{F}$ denote the set of joint distributions over the primitives of the model, $\mathbf{H}$ denote the set of joint distributions over the observables, and Γ be a collection of mappings $\gamma : \mathbf{F} \to \mathbf{H}$ representing the behaviors permitted under the model. A model $(\mathbf{F}, \Gamma)$ is identified if $\gamma(F) = \hat{\gamma}(\hat{F})$ implies $(F, \gamma) = (\hat{F}, \hat{\gamma})$

[1]eBay auctions typically have a very low minimum bid ("to stimulate bidding"), and a hidden reserve price below which the seller is not required to honor the winning bid.

for every $F, \hat{F} \in \mathbf{F}$ and every $\gamma, \hat{\gamma} \in \Gamma$. Intuitively, a model is identified if a set of observables could only have been generated by a single set of primitives under equilibrium behavior.

Two results from AH form the motivation for our work:

Result 1. *(AH Theorem 1) In the symmetric IPV model, the joint distribution of bidder values is identified from the transaction price of a second-price auction.*

Result 2. *(AH Theorem 4) In the symmetric PV model, the joint distribution of bidder values is not identified in a second-price auction unless all bids are observed.*

In practice, this means that when values are independent, the distribution of the second-highest bid in a second-price auction is sufficient to solve for the optimal values of the reserve price and entry fee. However, when values are not independent, multiple underlying joint distributions are consistent with any set of observations, and the optimal auction parameters may not be pinned down.

Haile and Tamer (2003, henceforth HT) point out that, in ascending auctions, bids cannot be interpreted as exact indications of bidders' values, since there is not a unique equilibrium bidding strategy. HT make two easily-justified assumptions about bidder behavior: that no bidder bids higher than her private value for the good, and that no losing bidder had a value higher than the winning bid (plus the minimum bid increment). These assumptions lead to upper and lower bounds on the distribution of bidder values given observed bid data; HT use these bounds to construct bounds on the revenue-maximizing reserve price as well. However, their work depends critically on the assumption that values are independent draws from the same distribution, or that, if they vary jointly, it is in response to covariates which are observed and understood by the seller. Haile and Tamer's other paper (in progress) examines the affiliated private values case, but looks only at bounding the distributions, not the optimal reserve price.

3 Our Environment

We assume a seller with one indivisible object to sell, which he values at v_0. He is committed to using a second-price sealed bid auction, and can set a reserve price and possibly an entry fee to maximize his expected payoff. We assume the number of potential bidders is fixed, and that each bidder knows her own value perfectly (private values) and will bid it, since this strategy is weakly dominant with private values. Whenever a positive entry fee is charged, we assume the bidders play an equilibrium where they enter when their expected surplus from the auction exceeds the entry fee, and those who enter still bid their value in the post-entry auction.

Formally, let n be the number of bidders, and $v_1, \ldots, v_n$ be their private values for the good. We assume the joint distribution $f(v_1, v_2, \ldots, v_n)$ is symmetric (exchangeable) but not necessarily independent, and has bounded support

$[\underline{v}, \overline{v}]^n$. Let $v^1 \geq v^2 \geq \cdots \geq v^n$ be the order statistics of the values.[2] Let $F_1(\cdot)$ and $F_2(\cdot)$ be the cumulative distribution functions[3] of v^1 and v^2, and f_1 and f_2 the corresponding densities. We assume the seller has knowledge of F_2, as from a series of transaction prices from similar auctions without reserve prices, but no other information about f. (Crucially, he does not know F_1.)

We will consider the range of optimal parameters in three cases. The first is when no entry fee can be used, so the seller need only select a reserve price. The second is when an entry fee can be used as well, and is paid *before* the bidders learn their valuations for the good; we will refer to these as *ex ante* entry fees. The third is when an entry fee can be used, and is paid *after* the bidders learn their valuations; we will refer to these as *ex post* entry fees.

4 Reserve Price Only

In the simplest case, the seller cannot charge an entry fee, so his only choice is the reserve price r. The seller's expected payoff, as a function of r, is

$$\Pr(v^1 \geq r > v^2)(r - v_0) + \Pr(v^2 \geq r)E(v^2 - v_0 | v^2 \geq r)$$

which we can rewrite as

$$\pi(r) = (r - v_0)(F_2(r) - F_1(r)) + \int_r^{\overline{v}} (v - v_0) dF_2(v)$$

Note that $\pi(v_0)$ is uniquely determined by F_2, so we can use this revenue level as a baseline.[4]

It is easy to show that $\pi(r) \leq \pi(v_0)$ when $r < v_0$ or $r > \overline{v}$; thus, we will always assume that the optimal reserve price lies in $[v_0, \overline{v}]$. We prove in the appendix that a maximizer always exists, that is, that $\pi(r)$ always attains a maximum on $[v_0, \overline{v}]$.[5]

4.1 Independent Private Values

If values are independently and identically distributed, AH Theorem 1 states that the joint distribution of bidder values is identified from F_2.[6] This result follows:

[2] Our notation differs from that in AH and HT, who use $v^{i:n}$ to indicate the i^{th} lowest of n values. Thus, v^1 in our notation is equivalent to $v^{n:n}$ in theirs.

[3] Cumulative distribution functions in this paper *exclude* any mass at the point being considered. That is, $F_i(r) \equiv \Pr(v^i < r)$, not $\Pr(v^i \leq r)$. This is done so that expected revenue can be written simply using these distribution functions.

[4] $\pi(r)$ can also be written as $E\left((\max(r, v^2) - v_0) \times \mathbf{1}_{v^1 \geq r}\right)$.

[5] π is not necessarily continuous in r, or the proof would be trivial. However, π is upper semicontinuous, meaning $\pi(r) \geq \lim \pi(r_n)$ as $r_n \to r$; this is sufficient to prove a maximum is attained over a compact space.

[6] If $H(v)$ is the cumulative density function of a single variable, then under independence,

$$F_2(v) = nH(v)^{n-1} - (n-1)H(v)^n$$

The function $B(x) = nx^{n-1} - (n-1)x^n$ is strictly increasing on $[0, 1]$ and therefore invertible; this inverse maps the observed distribution F_2 to the distribution of a single bidder value.

Theorem 1. *If values are distributed independently and symmetrically on $[\underline{v}, \overline{v}]$, then the optimal reserve price is identified from the distribution F_2 of the second-highest value.*

Since the joint distribution of bidder values is known, F_1 is known, so π is known; the problem reduces to maximizing a known function over the interval $[v_0, \overline{v}]$. Since we proved a maximizer exists, the result follows.

If values are i.i.d. $\sim H(\cdot)$, then $F_2(r) = nH(r)^{n-1} - (n-1)H(r)^n$ and $F_1(r) = H(r)^n$. When $H(\cdot)$ is differentiable, π is differentiable, and

$$\pi'(r) = F_2(r) - F_1(r) - (r - v_0)f_1(r) = nH^{n-1}(r)(1 - H(r) - (r - v_0)h(r))$$

This has the same sign as $1 - H(r) - (r - v_0)h(r)$, which is the first derivative of $(r - v_0)(1 - H(r))$. Thus, under pseudoconcavity, when the sign of π' is sufficient to find the maximizer, the seller's problem is the same as a monopolist's pricing problem with marginal cost v_0 and demand $1 - H(r)$.

4.2 Symmetric Private Values

AH Theorem 4 states that, with symmetric private values, the joint distribution is not identified from fewer than n order statistics. However, the proof is by construction of an arbitrarily small perturbation of the joint density function which leaves the observed order statistics unchanged; such a small shock may have little impact on a global policy variable such as optimal reserve price. We therefore give a stronger nonidentification result:

Theorem 2. *Take any observed distribution $F_2(\cdot)$ on $[\underline{v}, \overline{v}]$. For any value $r \in [v_0, \overline{v}]$, there exists a symmetric joint density f whose second order statistic has distribution F_2 and for which the optimal reserve price is r.*

We prove this result by constructing such a joint distribution. Let v be a random variable distributed $\sim F_2(\cdot)$. Define bidder values by

$$\begin{aligned} v_i &= \max\{r, v\} \\ v_{j \neq i} &= v \end{aligned}$$

with $i = \{1, 2, \ldots, n\}$ with equal probability. (That is, if $v \geq r$, all bidders have value $v_i = v$; if $v < r$, one bidder at random has value $v_i = r$, the rest have value $v_j = v$.) The second order statistic is always equal to v, and therefore has distribution F_2. As long as $r \geq v_0$, a reserve price of r maximizes revenue, since the bidder with the highest value always wins and pays his full value.

4.3 Two Cases of Positively Correlated Private Values

Two cases which have received attention in the literature are the cases of affiliated values and conditionally independent values. Affiliation is a strong local condition of positive association between variables; the technical condition is that the joint density function is log-supermodular.[7] We also continue to assume symmetry or exchangability, that is, that $f(v_1, \ldots, v_n) = f(\sigma(v_1, \ldots, v_n))$, where σ is an arbitrary permutation (reordering).

Conditionally independent values are best explained by example. A classic case is when bidder values are additively separable into "common" and "private" components, such as $v_i = v + \epsilon_i$, where ϵ_i are independent idiosyncratic shocks. For a given realization of v, the values v_i are then *i.i.d.* draws from a distribution which is conditional on v. This intuition extends to more general cases; conditionally independent values can be seen as *i.i.d.* draws from one of a number of different distributions, where the common distribution used is drawn randomly as well. Thus, a conditionally independent distribution can be defined as a set of distributions $\{H_\theta\}$ parametrized by a vector θ and a probability distribution for θ.

Without independent values, there are many possible joint distributions which are consistent with the observed distribution F_2 of the second order statistic. Thus, without some sort of prior over these distributions, we cannot pin down the expected revenue as a function of reserve price. What we can do is to characterize the range of possible values it could take, and the range of reserve prices which could potentially be optimal. (We use the same method as HT to bound the maximizer of a function from upper and lower bounds on the function.)

We prove two results. For any observed distribution F_2 and any value $r > v_0$, we show that the lower bound on $\pi(r)$ is below $\pi(v_0)$, and the upper bound on $\pi(r)$ is the value it would take under independence. These two results then lead to upper and lower bounds on the optimal reserve price.

Bounds on $\pi(r)$

With independent values, $\pi(r)$ was known; that is, the expected revenue given a reserve price was known, and could be maximized. Setting a reserve price above v_0 was still "risky," in the sense that for some realizations of $\{v_i\}$ it would preclude a profitable sale; but in an expectation sense, it was optimal.

With affiliated or conditionally independent values, a reserve price above v_0 is risky in a different sense. That is, without further information about the distribution of v^1, any reserve price above v_0 could potentially give lower expected profit than a reserve price of v_0.

Theorem 3. *For any observed distribution $F_2(\cdot)$ and any value $r > v_0$, there exists a joint density f of symmetric, affiliated, conditionally independent private values whose second-order statistic has distribution $F_2(\cdot)$ and such that $\pi(p) < \pi(v_0)$ for all $p \geq r$.*

The proof is in the appendix: we introduce a family of joint distributions f parameterized by a parameter σ, and show that for any p, $\pi(p) < \pi(v_0)$ when

[7] Or, $f(v)f(v') \leq f(v \vee v')f(v \wedge v')$, where $\vee$ and $\wedge$ are the componentwise max and min operators. See Milgrom and Weber (1982) for details.

σ is small enough. (As σ goes to zero, the values become arbitrarily highly correlated.)

Next, we show that $\pi(r)$ is weakly smaller than it would be under independence. To do this, we let H be the distribution such that the second-highest of n independent draws on H is $\sim F_2$; that is, define H implicitly by

$$F_2(v) = n(H(v))^{n-1} - (n-1)(H(v))^n$$

Let $F_1^I(v) = H(v)^n$, so F_1^I is the distribution of the first-order statistic of the independent joint density whose second-order statistic has distribution F_2. Finally, let $\pi^I(r) = (r - v_0)(F_2(r) - F_1^I(r)) + \int_r^{\overline{v}} (v - v_0) dF_2(v)$, so π^I is the expected revenue given independence.

Theorem 4. *If $\{v_i\}$ are symmetric and affiliated or conditionally independent, then $\pi(r) \leq \pi^I(r)$ for all $r \geq v_0$.*

The proof is in the appendix.

Bounds on Optimal Reserve Price

Let r^* be the true optimal reserve price. Since r^* maximizes π, $\pi(r^*) \geq \pi(v_0)$. Since $\pi(v_0)$ is known, this inequality, and bounds on $\pi(r)$, give bounds on the value of r^*.

Corollary 5. *For any observed distribution $F_2(\cdot)$, the optimal reserve price under affiliation or conditional independence could be arbitrarily close to v_0.*

Choose any $r > v_0$; Theorem 3 showed the existence of a joint distribution with $\pi(p) < \pi(v_0)$ above $p = r$, so by optimality, $r^* < r$. Taking r to v_0 proves the corollary. Thus, v_0 forms a tight lower bound on the optimal reserve price.

Corollary 6. *Let $\tilde{v}$ be the smallest value such that $\pi^I(r) < \pi(v_0)$ for $r > \tilde{v}$. Then if values are symmetric and affiliated or conditionally independent, $r^* \leq \tilde{v}$.*

If this were false, then $\pi(v_0) > \pi^I(r^*) \geq \pi(r^*)$, with the second inequality from Theorem 4. Since in general $\tilde{v} < \overline{v}$,[8] this gives an upper bound on r^*.

Unlike the lower bound, this upper bound may not be tight.[9] I still hope to find a proof that r^* must be less than or equal to the maximizer of π^I. Even without this, if we consider setting a reserve price as balancing the expected "upside" against the potential "downside", we can see Theorems 3 and 4 as statements that under symmetry and affiliation or conditional independence, a given reserve price $r > v_0$ has both less upside and more downside than it would under independence; on an intuitive level, this seems to favor a lower reserve price.

[8] Barring a large point mass in F_2 at $\overline{v}$.

[9] In fact, with affiliated values, if π^I is continuous and has a negative first derivative at $\tilde{v}$, then r^* is bounded away from $\tilde{v}$. However, the proof is tricky and not particularly enlightening.

5 Ex-Ante Entry Fees

Now suppose the seller can charge an entry fee, which is paid before the bidders learn their values. We are still committed to a second-price sealed-bid auction, and still assume that, after entry, all entrants bid their values. In this case (with ex-ante symmetry), it turns out to always be optimal to hold the efficient auction ($r = v_0$) and to extract all bidder surplus via the entry fee:

Lemma 7. *A reserve price of $r^* = v_0$ and an ex ante entry fee of*

$$e^* = \frac{1}{n} E \left(\max\{v^1, v_0\} - \max\{v^2, v_0\} \right)$$

maximize expected revenue.

The proof is in the appendix. Since the distribution of v^2, and therefore the value of $E \max(v^2, v_0)$, is known, this tells us that knowing (or bounding) the expectation of $\max\{v^1, v_0\}$ gives a solution (or bounds) for the optimal ex-ante entry fee. This leads to the following results:

Theorem 8. *If the seller can charge an ex ante entry fee e and a reserve price r, then the optimal reserve price is always $r^* = v_0$. Furthermore,*

- *if $\{v_i\}$ are independent, the optimal entry fee e^* is identified, call it e^I*
- *if $\{v_i\}$ are symmetric, then e^* could be any value within $[0, \overline{e}]$, with $\overline{e} > e^I$*
- *if $\{v_i\}$ are symmetric and affiliated or conditionally independent, then $e^* \in [0, e^I]$, with both bounds being tight.*

(To be clear, we are *not* trying to describe the optimal response to uncertainty, only to understand the range of parameters that could be optimal given some specific distribution consistent with our observables.)

Once the lemma has been proved, the theorem follows easily. In the case of independence, the distribution F_1 of v^1 is known, so we can explicitly calculate $E \max\{v^1, v_0\}$ and therefore e^I. In the case of symmetric affiliated or conditionally independent values, we proved that the true distribution F_1 is first-order stochastic dominated by F_1^I, the distribution of v^1 under independence; since $\max\{v^1, v_0\}$ is a nondecreasing function of v^1, its expectation is therefore weakly lower than under independence, leading to $e^* \leq e^I$; this bound must be tight since independence is a special case of both these conditions. For the lower bound, take the degenerate case where $v_i = v_j$ so $v^1 = v^2$ and $e^* = 0$. (For a nondegenerate case, we can use the construction in the proof of Theorem 3; as $\sigma \to 0$, $e^* \to 0$.)

Finally, for the symmetric case, let $\overline{e} = \frac{1}{n}(\overline{v} - E \max\{v^2, v_0\})$. Let v be a random variable distributed $\sim F_2(\cdot)$, and x be a random Boolean variable independent of v which takes the value 1 with probability s. Define bidder values by

$$\begin{aligned} v_i &= x\overline{v} + (1-x)v \\ v_{j \neq i} &= v \end{aligned}$$

with i chosen randomly among $\{1, \ldots, n\}$ with equal probability. Then (since x is independent of v)

$$\begin{aligned} e^* &= \tfrac{1}{n}E(\max\{v^1, v_0\} - \max\{v^2, v_0\}) \\ &= \tfrac{1}{n}E(x\overline{v} + (1-x)\max\{v, v_0\} - \max\{v, v_0\}) \\ &= \tfrac{1}{n}E(x(\overline{v} - \max\{v, v_0\})) \\ &= \tfrac{1}{n}E(x)E(\overline{v} - \max\{v, v_0\}) \\ &= \tfrac{1}{n}s(\overline{v} - E\max\{v^2, v_0\}) \\ &= s\overline{e} \end{aligned}$$

Varying s over $[0, 1]$ varies e^* over $[0, \overline{e}]$. To show $\overline{e}$ is the upper bound, note that a higher value of e^* would require $E(\max\{v^1, v_0\}) > \overline{v}$, which is impossible.

6 Ex-Post Entry Fees

Now suppose instead that the seller can charge an entry fee which is paid *after* the buyers learn their valuations. That is, the buyers learn v_i, then choose whether to pay the entry fee e, and then a second-price auction with reserve price r is held among those who entered.

MW (section 7) point out that when values are not independent, the entry decision of each bidder need not be increasing in her own private value – that is, a bidder may choose not to enter given a higher value than one at which she would enter. This makes it difficult to fully characterize optimal auction parameters. However, we offer the following partial result, which is clearly analogous to the lack of lower bounds on r^* (Corollary 5) and e^* (Theorem 8).

Theorem 9. *For any pair $(e, r) \neq (0, v_0)$ of an ex-post entry fee and a reserve price, and any observed distribution F_2, there exists a symmetric, affiliated, conditionally independent joint distribution f consistent with F_2 such that $\pi(e, r) < \pi(0, v_0)$.*

The proof is in the appendix. We assume that bidders play a (not necessarily symmetric) equilibrium in the entry game but do not publicly randomize, and those that enter bid their values. The intuition behind the proof is that, if values are highly correlated, then post-entry surplus is nearly zero if more than one bidder enters; thus, with a positive entry fee, bidders must enter with probability less than one regardless of their value. The possibility that nobody enters when values are high (above v_0) introduces an inefficiency that cannot be recaptured through the entry fee.

7 Conclusion

With independent private values, the optimal parameters for running a second-price auction are identified from the distribution of the transaction price, since this uniquely determines the joint distribution of bidder values; bounds on the distribution lead to bounds on the optimal parameters. With nonindependent private values, however, the optimal parameters are not identified. In particular, if the joint distribution of bidder values is assumed only to be symmetric, the optimal reserve price (when no entry fee is allowed) or the optimal ex-ante entry fee (when allowed) can take any value within a wide range. When symmetry and affiliation or conditional independence is assumed, the optimal reserve price (with no entry fee) is not bounded below away the seller's reserve value, but is bounded above. Similarly, the optimal entry fee, whether ex-post or ex-ante, is not bounded below away from 0. This suggests that high reserve prices or entry fees should be used more cautiously when bidder values are thought to be positively correlated; or, put another way, that parameters calculated assuming independence will tend to be biased upwards when values are actually positively correlated.

Appendix 1. Existence of a Maximizer

Claim. *For any joint distribution of bidder values on* $[\underline{v}, \overline{v}]^n$*, a reserve price exists which maximizes expected revenue.*

As we argued earlier, $\pi(r) \leq \pi(v_0)$ for $r \notin [v_0, \overline{v}]$; thus, a maximum for π over $[v_0, \overline{v}]$ is a global maximum.

A function $j : X \to \Re$ is **upper semicontinuous** (as a function, not a correspondence) if for any sequence $\{x_m\} \in X$ converging to $x \in X$,

$$j(x) \geq \lim_{m \to \infty} j(x_m)$$

(Of course, if j is continuous, then this condition is satisfied with equality.)

We prove our claim in two steps. First, we show that if X is a compact subset of the reals and $j : X \to \Re$ is upper semicontinuous and bounded above, then it achieves a maximum. Then, we show that $\pi : [v_0, \overline{v}] \to \Re$ is upper semicontinuous and bounded above and apply the first result.

Lemma 10. *Let* $X \subset \Re$ *be compact, and* $j : X \to \Re$ *be upper semicontinuous and bounded above. Then* j *achieves a maximum on* X*.*

Proof of Lemma. Since j is bounded above on X, a supremum exists, call it P. Let $\{\epsilon_m\}$ be any sequence of positive numbers converging to zero. For each $\epsilon_m > 0$, there must be some point $w_m \in X$ with $j(w_m) \geq P - \epsilon_m$. Now, any such sequence $\{w_m\}$ is a bounded (because X is bounded) sequence of real numbers; by the Bolzano-Weierstrass Theorem,[10] it has a convergent subsequence, call this subsequence $\{x_m\}$.

By construction, $j(w_m) \in [P - \epsilon_m, P]$; since $\epsilon_m \to 0$, $j(w_m) \to P$, so $j(x_m) \to P$. Since $\{x_m\}$ is a convergent subsequence, it has a limit point, call it x; since X is closed and $x_m \in X$ for all m, this limit point x is in X as well. By upper semicontinuity,

$$j(x) \geq \lim j(x_m) = P$$

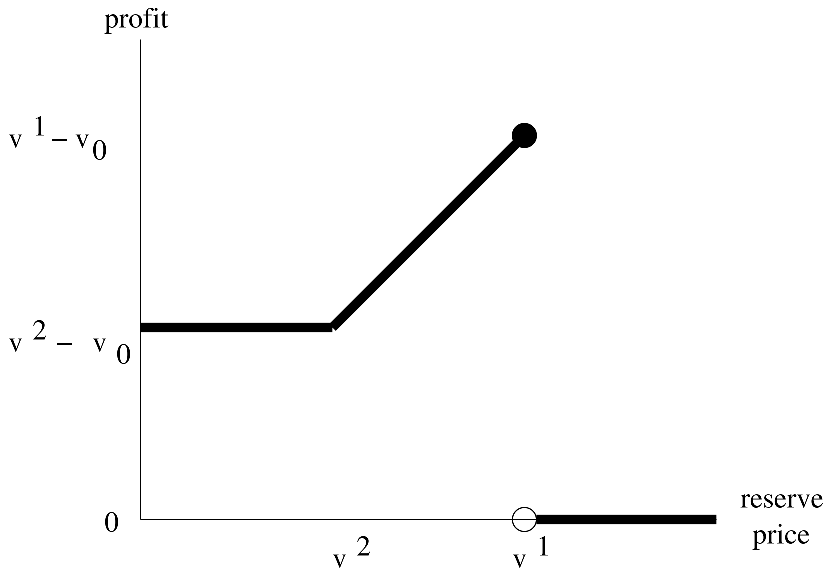
Since P is the supremum of j on X, $j(x) \leq P$, so $j(x) = P$; so the supremum P is attained at x, proving the lemma. □

Proof of Claim. For a given realization v^1, v^2 of the first and second order statistics of $\{v_i\}$, let $\pi_{v^1,v^2}(r)$ be the net profit of an auction with reserve price $r \in [v_0, \overline{v}]$. It is easy to see (as in the figure on the next page) that

$$\pi_{v^1,v^2}(r) = \begin{cases} v^2 - v_0 & \text{if } r \leq v^2 \\ r - v^0 & \text{if } v^2 < r \leq v^1 \\ 0 & \text{if } r > v^1 \end{cases}$$

(In the figure below, $v^2 > v_0$; if $v^2 \leq v_0$, only part of the graph represents $r \in [v_0, \overline{v}]$. The following argument holds in either case.)

[10] The Bolzano-Weierstrass Theorem states that any bounded sequence of real numbers has a convergent subsequence. An elegant proof can be found online at http://planetmath.org/encyclopedia/ProofOfBolzanoWeierstrassTheorem.html



By inspection, π_{v^1,v^2} is upper semicontinuous in r: it is continuous everywhere but at $r = v^1$; at $r = v^1$, it is equal to its limit from the left and greater than its limit from the right. (If $v^1 < v_0$, then $\pi(r) = 0$ on all of $[v_0, \overline{v}]$ and is therefore continuous.)

Expected profits are then the expectation of π_{v^1,v^2} over the joint distribution of v^1 and v^2, or

$$\pi(r) = \int \int \pi_{v^1,v^2}(r)\delta(v^1, v^2)dv^1dv^2$$

where $\delta(\cdot,\cdot)$ is the joint probability density of v^1 and v^2. (Such a δ exists for any joint distribution of values; we allow for point masses by allowing δ to be infinite, but still integrable, when necessary.)

To show π is upper semicontinuous, take any sequence $r_1, r_2, r_3, \ldots$ converging to r. We already showed that for any v^1 and v^2 and any $r \geq v_0$,

$$\pi_{v^1,v^2}(r) \geq \lim \pi_{v^1,v^2}(r_m)$$

and so

$$\pi_{v^1,v^2}(r)\delta(v^1, v^2) \geq \lim \pi_{v^1,v^2}(r_m)\delta(v^1, v^2)$$

since $\delta \geq 0$. We integrate both sides over the support of $\delta(\cdot,\cdot)$ and move the limit outside the integral (since absolute convergence holds), giving $\pi(r) \geq \lim_{m\to\infty} \pi(r_m)$, proving π is upper semicontinuous on $[v_0, \overline{v}]$.

We can bound π above by $\overline{v} - v_0$.[11] Then since $X = [v_0, \overline{v}] \subset \Re$ is compact and π is bounded above and upper semi continuous, π attains a maximum. As we argued above, a maximum over $[v_0, \overline{v}]$ is sufficient. □

[11] Since $r \leq \overline{v}$, $\pi(r) = (r - v_0)(F_2(r) - F_1(r)) + \int_r^{\overline{v}} (v - v_0)dF_2(v)dv \leq$ $(\overline{v} - v_0)(F_2(r) - F_1(r)) + (\overline{v} - v_0)(1 - F_2(r)) = (\overline{v} - v_0)(1 - F_1(r)) \leq \overline{v} - v_0$

Appendix 2. Proof of Theorem 3

Claim. *For any F_2 and any $r > v_0$, there exists a symmetric, affiliated, conditionally independent joint distribution f consistent with F_2 such that $\pi(p) < \pi(v_0)$ for all $p \geq r$.*

Proof. Let $v, \epsilon_1, \epsilon_2, \ldots, \epsilon_n$ be independent random variables distributed $\sim U[0,1]$. For a given value of σ, let $J_\sigma(\cdot)$ be the distribution of $v + \sigma\epsilon^2$, where ϵ^2 is again the second-largest value of $\{\epsilon_i\}$. Let the bidders' values be

$$v_i = F_2^{-1}(J_\sigma(v + \sigma\epsilon_i))$$

where $F_2^{-1}(y)$ is defined as $\sup\{x : F_2(x) \leq y\}$ if F_2 is not strictly increasing and continuous. We claim that these values are symmetric, affiliated, and conditionally independent, that $v^2 \sim F_2(\cdot)$, and that, by making σ small, we can force $\pi(p)$ to be negative above any $r > v_0$.

Properties of $\{v_i\}$

By construction, bidder values are symmetric and conditionally independent – for a given realization of v, the v_i are all *i.i.d.* draws from the same (conditional) distribution. Next, we show the v_i are affiliated. Let $x_i = v + \sigma\epsilon_i$. Since $v_i = (F_2^{-1} \circ J_\sigma)(x_i)$ and $F_2^{-1} \circ J_\sigma$ is nondecreasing, Milgrom and Weber (henceforth MW) Theorem 3 states that $\{v_i\}$ are affiliated if $\{x_i\}$ are affiliated. Adopting the notation of MW section 3 (for this argument only), we rewrite the joint probability density function of $v, x_1, \ldots, x_n$ as

$$f(v, x_1, \ldots, x_n) = h(v)g(x_1|v) \cdots g(x_n|v)$$

Theorem 1 of MW states that a density which is the product of affiliated functions is affiliated; $h(\cdot)$ is trivially affiliated (since it only takes one variable), so f is affiliated (that is, $v, x_1, \ldots, x_n$ are affiliated) as long as $g(x_i|v)$ satisfies the affiliation inequality

$$g(x_i|v)g(x_i'|v') \geq g(x_i'|v)g(x_i|v')$$

for all $v' > v, x_i' > x_i$. Now, since $g(x_i|v) = j\left(\frac{x_i - v}{\sigma}\right)$ (with $j(\cdot)$ the density of ϵ_i), this is the same condition as $\log j(\cdot)$ being concave (with $\log 0 \equiv -\infty$). (If $\log j(\cdot)$ is concave, this implies $\log j(z+\delta) - \log j(z)$ is decreasing in z; let $\delta = \frac{v'-v}{\sigma}$, increase z from $\frac{x_i - v'}{\sigma}$ to $z' = \frac{x_i' - v'}{\sigma}$, and the affiliation inequality follows.) Since ϵ_i is distributed $U[0,1]$ and the density $j(x) = \mathbf{1}_{x \in [0,1]}$ is log-concave, $\{v, x_1, \ldots, x_n\}$ are affiliated, so (using Theorems 3 and 4 from MW) $\{x_i\}$ and therefore $\{v_i\}$ are affiliated.

Distribution of the Second Order Statistic

Next, we show that these values are consistent with F_2, that is, that the second-highest bidder value v^2 is indeed distributed $\sim F_2$. Specifically, we need to show that

$$\Pr(v^2 < r) = F_2(r)$$

From its definition, $F_2^{-1}(y)$ is an increasing function of y, since it is the supremum of a set which grows as y increases. J_σ is a distribution function, and therefore increasing, so $F_2^{-1} \circ J_\sigma$ is increasing function; so for any realization of $\{\epsilon_i\}$, the second-order statistic of v_i is equal to $F_2^{-1}(J_\sigma(v + \sigma\epsilon^2))$. (That is, $F_2^{-1} \circ J_\sigma(v + \sigma\epsilon)$ takes the second-highest of the ϵ_i to the second-highest of the v_i). Thus,

$$\Pr(v^2 < r) = \Pr(F_2^{-1}(J_\sigma(v + \sigma\epsilon^2)) < r)$$

Now, $F_2^{-1}(y) < x$ if and only if $y < F_2(x)$,[12] so

$$\Pr(v^2 < r) = \Pr(J_\sigma(v + \sigma\epsilon^2) < F_2(r))$$

Since $v + \sigma\epsilon^2 \sim J_\sigma$ (this is how J_σ was defined), it follows[13] that $J_\sigma(v + \sigma\epsilon^2) \sim U[0,1]$, so the probability it is less than $F_2(r)$ is simply $F_2(r)$, proving that $\Pr(v^2 < r) = F_2(r)$.

Making π Small

Finally, we claim that for any $r > v_0$, there exists a $\sigma > 0$ such that $\pi(p) < \pi(v_0)$ for all $p \geq r$. To prove this, note that

$$\begin{aligned}
\Pr(v^1 \geq p > v^2) &= \Pr\left(F_2^{-1}(J_\sigma(v + \sigma\epsilon^1)) \geq p > F_2^{-1}(J_\sigma(v + \sigma\epsilon^2))\right) \\
&= \Pr\left(v + \sigma\epsilon^1 \geq J_\sigma^{-1}(F_2(p)) > v + \sigma\epsilon^2\right) \\
&\leq \Pr\left(v \in [J_\sigma^{-1}(F_2(p)) - \sigma, J_\sigma^{-1}(F_2(p))]\right) \\
&\leq \sigma
\end{aligned}$$

(The first equality is a definition; the next uses the properties of F_2^{-1} (see footnote) and the fact that J_σ is invertible with an increasing inverse; the first inequality is because $\epsilon^1 \leq 1$ and $\epsilon^2 \geq 0$, and the last because $v \sim U[0,1]$.) Now, set

$$\sigma < \frac{1}{\overline{v} - v_0} \int_{v_0}^{r} (v - v_0) dF_2(v) dv$$

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[12] Since $F_2(x) = \Pr(v^2 < x)$, F_2 must be lower semicontinuous: as $x_n \nearrow x$, $F_2(x) - F_2(x_n) = \Pr(x_n \leq v^2 < x) \to 0$. (This is trivial if F_2 has finitely many point masses: eventually, x_n must be closer to x than the nearest point mass. If F_2 has infinitely many point masses, they must still be countable; sort them in decreasing order and call μ_k the mass of the k^{th} highest one. Let $R_n = \sum_{k \geq n} \mu_k$; since the total sum is bounded ($R_1 \leq 1$) and the sequence is decreasing, the partial sum $R_n \to 0$. Now, let $\hat{x}_k$ be greater than $x - 1/k$ and big enough so that $[\hat{x}_k, x)$ does not contain any of masses 1 through k. Then the sum of the weights on the point masses in $[\hat{x}_k, x)$ is $\leq R_k$ and therefore goes to zero, so the total probability weight goes to zero as well.) Since F_2 is lower semicontinuous, its lower contour sets are closed, so the supremum is attained; then

$$x > F_2^{-1}(y) \longleftrightarrow x > \sup\{\hat{x} : F_2(\hat{x}) \leq y\} = \max\{\hat{x} : F_2(\hat{x}) \leq y\} \longleftrightarrow F_2(x) > y$$

Also, since $z \geq x$ if and only if $z \not< x$, $F_2^{-1}(y) \geq x \longleftrightarrow y \geq F_2(x)$, which we will require shortly. Note that it is *not* necessarily true that $F_2^{-1}(y) > x \longleftrightarrow y > F_2(x)$.

[13] If $x \sim Z(\cdot)$ and Z is invertible, then $Z(x) \sim U[0,1]$, since for $y \in [0,1]$,

$$\Pr(Z(x) \leq y) = \Pr(x \leq Z^{-1}(y)) = Z(Z^{-1}(y)) = y$$

If $p > \overline{v} > v_0$, then $\pi(p) = 0 < \pi(v_0)$. Otherwise,

$$\begin{array}{rcl} \pi(p) - \pi(v_0) & = & (p - v_0)\Pr(v^1 \geq p > v^2) - \int_{v_0}^{p}(v - v_0)dF_2(v) \\ & \leq & (p - v_0)\sigma - \int_{v_0}^{p}(v - v_0)dF_2(v) \\ & < & \frac{p - v_0}{\overline{v} - v_0}\int_{v_0}^{r}(v - v_0)dF_2(v)dv - \int_{v_0}^{p}(v - v_0)dF_2(v) \\ & \leq & \int_{v_0}^{r}(v - v_0)dF_2(v) - \int_{v_0}^{p}(v - v_0)dF_2(v) \\ & \leq & -\int_{r}^{p}(v - v_0)dF_2(v)dv \\ & \leq & 0 \end{array}$$

since $\overline{v} \geq p \geq r > v_0$, so $\pi(p) < \pi(v_0)$. □

Appendix 3. Proof of Theorem 4

Claim. *Under symmetry and affiliation or conditional independence,* $\pi(r) \leq \pi^I(r)$ *if* $r \geq v_0$.

Proof is in three steps:

- $F_1(r) \geq F_1^I(r)$ if values are symmetric and affiliated
- $F_1(r) \geq F_1^I(r)$ if values are conditionally independent
- $\pi(r) \leq \pi^I(r)$ if $F_1(r) \geq F_1^I(r)$ and $r \geq v_0$

1. If $\{v_i\}$ are symmetric and affiliated, $F_1(r) \geq F_1^I(r)$ for all r.

Fix r. Choose arbitrary $i \in \{0, 1, \ldots, n-2\}$. Let X and Y be shorthand for the following statements:

$$X = \text{``}v_1, \ldots, v_i \geq r,\;\; v_{i+1}, \ldots, v_{n-2} < r,\;\; v_{n-1} < r\text{''}$$

$$Y = \text{``}v_1, \ldots, v_i \geq r,\;\; v_{i+1}, \ldots, v_{n-2} < r,\;\; v_{n-1} \geq r\text{''}$$

Under affiliation,

$$\Pr(v_n \geq r|X) \leq \Pr(v_n \geq r|Y)$$

because $\mathbf{1}_{v_n \geq r}$ is an increasing function of v_n, and therefore its expectation is nondecreasing in the values of the other v_j (MW Theorem 5). Of course, this also means $\Pr(v_n < r|X) \geq \Pr(v_n < r|Y)$, and so

$$\frac{\Pr(X)\Pr(v_n \geq r|X)}{\Pr(X)\Pr(v_n < r|X)} \leq \frac{\Pr(Y)\Pr(v_n \geq r|Y)}{\Pr(Y)\Pr(v_n < r|Y)}$$

or

$$\frac{\Pr(X, v_n \geq r)}{\Pr(X, v_n < r)} \leq \frac{\Pr(Y, v_n \geq r)}{\Pr(Y, v_n < r)}$$

Now, let P_i be the (true) probability that exactly i bidders have values greater than or equal to r, and P_i^I be the probability under independently distributed values consistent with F_2. Recall that if X holds then i of the first $n-1$ values are above r, and if Y holds then $i+1$ are. By symmetry,

$$\begin{aligned}
P_i &= {}_nC_i & \Pr(X, v_n < r) \\
P_{i+1} &= {}_nC_{i+1} & \Pr(X, v_n \geq r) \\
&= {}_nC_{i+1} & \Pr(Y, v_n < r) \\
P_{i+2} &= {}_nC_{i+2} & \Pr(Y, v_n \geq r)
\end{aligned}$$

and so our inequality becomes

$$\frac{\frac{1}{{}_nC_{i+1}} P_{i+1}}{\frac{1}{{}_nC_i} P_i} \leq \frac{\frac{1}{{}_nC_{i+2}} P_{i+2}}{\frac{1}{{}_nC_{i+1}} P_{i+1}}$$

Also, note that if the values are independent, then by definition $Pr(v_n \geq r)$ does not depend on the value of v_{n-1}; then the same inequalities all hold with equality, so

$$\frac{\frac{1}{{}_nC_{i+1}} P^I_{i+1}}{\frac{1}{{}_nC_i} P^I_i} = \frac{\frac{1}{{}_nC_{i+2}} P^I_{i+2}}{\frac{1}{{}_nC_{i+1}} P^I_{i+1}}$$

By definition,

$$F_1(r) = P_0, \quad F^I_1(r) = P^I_0, \quad F_2(r) = P_0 + P_1 = P^I_0 + P^I_1$$

Suppose that $F_1(r) < F^I_1(r)$, so $P_0 < P^I_0$. It must be that $P_1 > P^I_1$, since $P_0 + P_1 = P^I_0 + P^I_1 = F_2(r)$. Then

$$\frac{\frac{1}{{}_nC_2} P_2}{\frac{1}{{}_nC_1} P_1} \geq \frac{\frac{1}{{}_nC_1} P_1}{\frac{1}{{}_nC_0} P_0} > \frac{\frac{1}{{}_nC_1} P^I_1}{\frac{1}{{}_nC_0} P^I_0} = \frac{\frac{1}{{}_nC_2} P^I_2}{\frac{1}{{}_nC_1} P^I_1}$$

so $\frac{P_2}{P_1} > \frac{P^I_2}{P^I_1}$; since $P_1 > P^I_1$, this means $P_2 > P^I_2$. It similarly follows that $\frac{P_3}{P_2} > \frac{P^I_3}{P^I_2}$, so $P_3 > P^I_3$, and similarly $P_4 > P^I_4$, etc. Since we knew at the outset that $P_0 + P_1 = P^I_0 + P^I_1$, this means

$$P_0 + P_1 + P_2 + \ldots + P_n > P^I_0 + P^I_1 + P^I_2 + \ldots + P^I_n$$

which is impossible since each side must sum to 1. This contradiction proves that $F_1(r) \geq F^I_1(r)$.

2. If $\{v_i\}$ are conditionally independent, $F_1(r) \geq F^I_1(r)$ for all r.

Define $A, B : [0,1] \to [0,1]$ by $A(x) = x^n$ and $B(x) = nx^{n-1} - (n-1)x^n$. For a given distribution $H(\cdot)$ of one variable, then, $A(H(\cdot))$ and $B(H(\cdot))$ are the distributions of the first and second order statistic, respectively, of n independent draws on H.

If values are conditionally independent, let $\{H_\theta\}$ be the set of distributions from which values may be independently drawn. It is easy to show that

$$F_2(r) = E_\theta B(H_\theta(r)) \quad \text{and} \quad F_1(r) = E_\theta A(H_\theta(r))$$

Furthermore, if values were actually independently distributed according to a distribution H, then $F_2(r) = B(H(r))$; then $H(r) = B^{-1}(F_2(r))$ under the assumption of independence, meaning

$$F_1^I(r) = A\left(B^{-1}(F_2(r))\right) = (A \circ B^{-1})E_\theta B(H_\theta(r))$$

Let $\xi \equiv A \circ B^{-1}$. We will show $\xi : [0,1] \to [0,1]$ is convex.[14] Then by Jensen's inequality,

$$\xi\left(E_\theta\{B(H_\theta(r))\}\right) \leq E_\theta\{\xi(B(H_\theta(r)))\}$$

which is equivalent to

$$F_1^I(r) \leq E_\theta(A \circ B^{-1})(B(H_\theta(r))) = E_\theta A(H_\theta(r)) = F_1(r)$$

To show ξ is convex, we calculate its second derivative. The fact that $(B^{-1})' = \frac{1}{B'}$, $(B^{-1})'' = -\frac{B''}{(B')^3}$, and the chain rule give

$$\xi''(F_2(r)) = \frac{1}{(B')^3}\left(A''B' - A'B''\right)$$

with all terms on the right evaluated at $B^{-1}(F_2(r))$. Letting $H = B^{-1}(F_2(r))$, plugging in the derivatives of A and B, and simplifying gives

$$\xi''(F_2) = \frac{1}{(B')^3} n^2(n-1)H^{2n-4}$$

which, since $B' = n(n-1)H^{n-2}(1-H) \geq 0$, is itself greater than or equal to zero. Since $\xi'' \geq 0$, ξ is convex and the result follows.

3. If $F_1(r) \geq F_1^I(r)$ and $r \geq v_0$ then $\pi(r) \leq \pi^I(r)$.

From their definitions,

$$\pi^I(r) - \pi(r) = (r - v_0)(F_1(r) - F_1^I(r))$$

so for $r \geq v_0$, $F_1 \geq F_1^I \to \pi \leq \pi^I$. Since the first two claims establish that $F_1 \geq F_1^I$, this proves the theorem. □

[14] It is tempting to see A, B, and ξ as functions defined on the space of distributions; note instead that all three are defined pointwise from $[0,1]$ to $[0,1]$, which makes convexity simpler.

Appendix 4. Proof of Lemma 7

Claim. *A reserve price of* $r^* = v_0$ *and an ex ante entry fee of*

$$e^* = \frac{1}{n} E\left(\max\{v^1, v_0\} - \max\{v^2, v_0\}\right)$$

maximize expected revenue.

Proof. First, suppose that bidders do not randomize entry, so they play an asymmetric equilibrium where each entrant has nonnegative expected payoff. For a given reserve price r and entry fee e, let $j \le n$ be the number of bidders who choose to pay the entry fee and participate. For them to enter, it must be that

$$e \le \frac{1}{j} E\left((v^{j:j} - \max\{r, v^{j-1:j}\}) \times \mathbf{1}_{v^{j:j} \ge r}\right)$$

since the right-hand side is equal to each entrant's expected surplus from the auction given j bidders. (Recall that $v^1 = v^{n:n}$.) The expected revenue from the auction (aside from the entry fees collected) is

$$U = E\left((\max\{r, v^{j-1:j}\} - v_0) \times \mathbf{1}_{v^{j:j} \ge r}\right)$$

so the overall seller surplus (assuming that e is set as high as it can be) is $je + U =$

$$E\left((v^{j:j} - \max\{r, v^{j-1:j}\}) \times \mathbf{1}_{v^{j:j} \ge r}\right) + E\left((\max\{r, v^{j-1:j}\} - v_0) \times \mathbf{1}_{v^{j:j} \ge r}\right)$$

$$= E\left((v^{j:j} - v_0) \times \mathbf{1}_{v^{j:j} \ge r}\right)$$

which is maximized when $j = n$ and $r = v_0$. Thus, the optimal reserve price is $r^* = v_0$, and the optimal entry fee allows n entrants, so

$$e^* = \frac{1}{n} E\left((v^1 - \max\{v_0, v^2\}) \times \mathbf{1}_{v^1 \ge v_0}\right) = \frac{1}{n} E\left(\max\{v^1, v_0\} - \max\{v^2, v_0\}\right)$$

Next, suppose instead that bidders randomize symmetrically, that is, given (e, r), each bidder enters with probability s. In order to randomize, bidders must be indifferent, so

$$e = E_j \frac{1}{j}\left((v^{j:j} - \max\{r, v^{j-1:j}\}) \times \mathbf{1}_{v^{j:j} \ge r}\right)$$

where j now takes values in $\{0, 1, \ldots, n\}$ according to a binomial distribution. Expected revenue is then

$$E_j(je + U(j))$$

where $U(j)$ is the post-entry expected profit generated by the auction with j entrants and reserve price r. This equals

$$E_j j \times E_j \left\{\frac{1}{j} E_v \left((v^{j:j} - \max\{r, v^{j-1:j}\}) \times \mathbf{1}_{v^{j:j} \ge r}\right)\right\} +$$

$$E_j E_v \left((\max(r, v^{j-1:j}) - v_0) \times \mathbf{1}_{v^{j:j} \geq r} \right)$$

Since the term in curly-brackets is decreasing in j, this is less than or equal to

$$E_j \left\{ \frac{j}{j} E_v \left((v^{j:j} - \max\{r, v^{j-1:j}\}) \times \mathbf{1}_{v^{j:j} \geq r} \right) \right\} +$$

$$E_j E_v \left((\max(r, v^{j-1:j}) - v_0) \times \mathbf{1}_{v^{j:j} \geq r} \right)$$
$$= E_j E_v \left((v^{j:j} - v_0\}) \times \mathbf{1}_{v^{j:j} \geq r} \right)$$
$$\leq E_v \left((v^{n:n} - v_0\}) \times \mathbf{1}_{v^{n:n} \geq v_0} \right)$$

so it is always optimal to set j uniformly equal to n (so $s = 1$) and $r = v_0$. Returning to the constraint on e gives $e = e^*$ as before. □

Appendix 5. Proof of Theorem 9

Claim. *Suppose bidders play a symmetric entry strategy. For any pair* $(e, r) \neq (0, v_0)$ *of an ex-post entry fee and a reserve price, and any observed distribution* F_2*, there exists a symmetric, affiliated, conditionally independent joint distribution* f *consistent with* F_2 *such that* $\pi(e, r) < \pi(0, v_0)$.

Proof. We will use the degenerate joint distrubtion where all bidder values are identical, that is, $v_i = v_j = v$, with v distributed according to the observed distribution F_2; this distribution works for *any* (e, r). For a nondegenerate example, we could use the same construction as that in the proof of theorem 3; for any given (e, r), this construction should work for σ adequately small.

We assume potential entrants play an equilibrium entry strategy, and bid their values if they enter. Note that if $e = 0$, everyone enters, and the expected revenue is

$$\int_r^{\overline{v}} (v - v_0) dF_2(v)$$

which is less than $\int_{v_0}^{\overline{v}} (v - v_0) dF_2(v) = \pi(0, v_0)$ if $r \neq v_0$. Furthermore, if $e + r > \overline{v}$, nobody will ever enter, since even with the highest possible value and no other entrants, entering and winning the auction at the reserve price gives negative surplus, so then $\pi(e, r) = 0 < \pi(0, v_0)$. Thus, for the remainder of the proof, we will assume that $e > 0$ and $e + r < \overline{v}$.

First, suppose that potential entrants play a symmetric entry strategy. The entry strategy takes the form $\tau : [\underline{v}, \overline{v}] \to [0, 1]$, where $\tau(v)$ represents the probability with which each bidder enters given a realization v of his private value. Since post-entry surplus is zero if more than one bidder enters (since the price paid will then be v), equilibrium behavior requires that if $\tau(v) > 0$ for a given value v, then

$$e \leq (1 - \tau(v))^{n-1} (v - r)$$

(The right-hand side is the expected post-entry surplus given entry, since with probability $(1 - \tau(v))^{n-1}$, nobody else will enter.)

Let $\pi_v(e,r)$ denote the seller's expected revenue for a given realization of v, so that $\pi(e,r) = E_v \pi_v(e,r)$. Now, (letting τ denote $\tau(v)$)

$$\pi_v(e,r) = ne\tau + n\tau(1-\tau)^{n-1}(r - v_0) + \left(1 - (1-\tau)^n - n\tau(1-\tau)^{n-1}\right)(v - v_0)$$

since with probability $n\tau(1-\tau)^{n-1}$, one bidder will enter (leading to a sale at the reserve price), and with probability $1-(1-\tau)^n - n\tau(1-\tau)^{n-1}$, more than one bidder will enter (leading to a sale at price v). Note that if $\tau = 0$ then $\pi_v(e,r) = 0$; if not, we can plug in the previous inequality, and so

$$\pi_v(e,r) \leq n\tau(1-\tau)^{n-1}(v - r) + n\tau(1-\tau)^{n-1}(r - v_0) +$$
$$\left(1 - (1-\tau)^n - n\tau(1-\tau)^{n-1}\right)(v - v_0)$$

Rearranging,

$$\pi_v(e,r) \leq n\tau(1-\tau)^{n-1}\left((v-r) + (r - v_0) - (v - v_0)\right) + \left(1 - (1-\tau)^n\right)(v - v_0)$$
$$= \left(1 - (1-\tau)^n\right)(v - v_0)$$

Note that if $\tau = 0$, this last term equals 0, so in either case ($\tau = 0$ or $\tau > 0$),

$$\pi_v(e,r) \leq \left(1 - (1-\tau(v))^n\right)(v - v_0)$$

Now, rearranging the earlier equilibrium condition, if $\tau(v) > 0$, it must be that $(1-\tau(v))^{n-1} \geq \frac{e}{v-r}$. Since $v \leq \overline{v}$, this means that

$$(1-\tau(v))^{n-1} \geq \frac{e}{\overline{v} - r}$$

and therefore that

$$(1-\tau(v))^n \geq \left(\frac{e}{\overline{v} - r}\right)^{\frac{n}{n-1}}$$

(Recall that we assumed earlier that $e + r \leq \overline{v}$ and therefore the right-hand side is not greater than 1.) So

$$\pi_v(e,r) \leq \max\left\{0, \left(1 - \left(\frac{e}{\overline{v} - r}\right)^{\frac{n}{n-1}}\right)(v - v_0)\right\}$$

which means that

$$\pi(e,r) = \int \pi_v(e,r)dF_2(v) \leq \left(1 - \left(\frac{e}{\overline{v} - r}\right)^{\frac{n}{n-1}}\right)\int_{v_0}^{\overline{v}}(v - v_0)dF_2(v)$$
$$= \left(1 - \left(\frac{e}{\overline{v} - r}\right)^{\frac{n}{n-1}}\right)\pi(0, v_0) < \pi(0, v_0)$$

If the equilibrium played in the entry game is not symmetric, the algebra is messier but the logic is the same. We do not allow for correlated equilibrium, so each player has an entry strategy $\tau_i : [\underline{v}, \overline{v}] \rightarrow [0,1]$. This time, the probability that one bidder enters (given a realization of v) is

$$\sum_i \left(\tau_i(v) \prod_{j \neq i} (1 - \tau_j(v)) \right)$$

and the probability that nobody enters is $\prod_i (1 - \tau_i(v))$, so

$$\pi_v(e,r) = \sum_i \tau_i(v) e + (r - v_0) \sum_i \left(\tau_i(v) \prod_{j \neq i} (1 - \tau_j(v)) \right) +$$

$$(v - v_0) \left(1 - \prod_i (1 - \tau_i(v)) - \sum_i \left(\tau_i(v) \prod_{j \neq i} (1 - \tau_j(v)) \right) \right)$$

Equilibrium play requires that if $\tau_i(v) > 0$,

$$e \leq (v - r) \prod_{j \neq i} (1 - \tau_j(v))$$

and so in either case (whether $\tau_i(v) = 0$ or not)

$$\tau_i(v) e \leq (v - r) \tau_i(v) \prod_{j \neq i} (1 - \tau_j(v))$$

Plugging this into our equation for $\pi_v(e,r)$ gives

$$\pi_v(e,r) \leq \sum_i \left(\tau_i(v) \prod_{j \neq i} (1 - \tau_j(v)) \right) ((v - r) + (r - v_0) - (v - v_0)) +$$

$$\left(1 - \prod_i (1 - \tau_i(v)) \right) (v - v_0) = \left(1 - \prod_i (1 - \tau_i(v)) \right) (v - v_0)$$

Now, for a given realization of v, let k be the identity of the potential entrant with the *second-highest* value of $\tau_i(v)$ (choose any in the event of a tie). If $\tau_k(v) = 0$, then only one bidder considers entering, knowing that all the others will not; he enters if $e + r \leq v$, and so

$$\pi_v(e,r) = (e + r - v_0) \times \mathbf{1}_{e+r \leq v}$$

On the other hand, if $\tau_k(v) > 0$, then

$$e \leq (v - r) \prod_{j \neq k} (1 - \tau_j(v))$$

and so

$$\prod_{j \neq k} (1 - \tau_j(v)) \geq \frac{e}{v - r} \geq \frac{e}{\overline{v} - r}$$

Now, since $\prod_{j \neq k}(1-\tau_j(v)) \geq \frac{e}{\overline{v}-r}$, it must be that $1-\tau_j(v) \geq \frac{e}{\overline{v}-r}$ for every $j \neq k$. But since $\tau_k(v)$ is not the largest of the $\tau_i(v)$, this means $1-\tau_k(v) \geq \frac{e}{\overline{v}-r}$ as well, so

$$\prod_i (1-\tau_i(v)) \geq \left(\frac{e}{\overline{v}-r}\right)^2$$

So we've proven that for any v, either

$$\pi_v(e,r) = (e+r-v_0) \times \mathbf{1}_{e+r \leq v} \quad (\leq v - v_0)$$

or

$$\pi_v(e,r) \leq \left(1 - \left(\frac{e}{\overline{v}-r}\right)^2\right)(v-v_0)$$

Now, consider two cases. In the first, the measure (with respect to F_2) of $v > v_0$ such that the second inequality holds is positive; in the second, it is zero. In the first case,

$$\pi(e,r) \leq \int_{v_0}^{\overline{v}} (v-v_0) dF_2(v) - \left(\frac{e}{\overline{v}-r}\right)^2 \int (v-v_0)\delta(v) dF_2(v)$$

where $\delta(v)$ is an indicator function for whether the second inequality holds; since $\delta(v) = 1$ over a part of $(v_0, \overline{v}]$ with positive measure, the second term is strictly positive, and so

$$\pi(e,r) < \int_{v_0}^{\overline{v}} (v-v_0) dF_2(v) = \pi(0, v_0)$$

In the other case,

$$\pi(e,r) = \int_{e+r}^{\overline{v}} (e+r-v_0) dF_2(v) < \int_{e+r}^{\overline{v}} (v-v_0) dF_2(v) \leq \pi(0, v_0)$$

so long as F_2 puts positive weight on $(v_0, \overline{v}]$. □

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