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Efficiency with Endogenous Population Growth

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Efficiency with Endogenous Population Growth*

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Abstract

In this paper, we generalize the notion of Pareto-efficiency to make it applicable to environments with endogenous populations. Two efficiency concepts are proposed, $\mathcal{P}$-efficiency and $\mathcal{A}$-efficiency. The two concepts differ in how they treat potential agents that are not born. We show that these concepts are closely related to the notion of Pareto-efficiency when fertility is exogenous. We then prove versions of the first welfare theorem assuming that decision making is efficient within the dynasty. We discuss two sets of sufficient conditions for noncooperative equilibria of family decision problems to be efficient. These include the Barro and Becker model as a special case. Finally, we study examples of equilibrium settings in which fertility decisions are not efficient, and classify them into ones where inefficiencies arise inside the family and ones where they arise across families.

Keywords: Pareto optimality, first welfare theorem, fertility, dynasty, altruism

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1 Introduction

Interest in the determinants of the equilibrium path for population has increased recently. (See Becker and Barro (1988), Barro and Becker (1989), Raut (1990), Doepke (2001), Fernandez-Villaverde (2001), Boldrin and Jones (2002), and Tertilt (2004). See Nerlove and Raut (1997) for a survey.) Surprisingly, little of this literature has used the tools of modern welfare economics (for example, Debreu (1962)) to address the normative questions that arise. This is because, at least in part, the usual notion of Pareto-efficiency is not well defined for environments in which the population is endogenous. To illustrate this, consider the following example. Compare an allocation with two agents, each consuming one unit of a lone consumption good, with an allocation where only one agent is born, but consumes two units of the consumption good. Is one allocation Pareto-superior to the other? Pareto-efficiency would involve a comparison, for each person, of the two allocations. But since different sets of people are alive in the two allocations, such a person-by-person comparison seems impossible.

In this paper, we generalize the notion of Pareto-efficiency to make it applicable to environments with endogenous populations. We propose two new efficiency concepts: $\mathcal{P}$-efficiency and $\mathcal{A}$-efficiency. These differ in the way that potential agents that are not born are treated. In the first, $\mathcal{P}$-efficiency, unborn children are treated symmetrically with the born agents (i.e., they have utility functions etc.), but with a limited choice set.[1] In the second, $\mathcal{A}$-efficiency, efficiency is defined only through comparisons among agents that are born (and hence it is not necessary that the unborn have well defined utility functions). We show that these two concepts are closely related to the

[1]Throughout, we do not take a stand on how to evaluate the utility of the unborn. Such a task is well beyond the scope of this paper. Rather, we propose two alternative definitions of Pareto-optimality which are at opposite extremes of the spectrum of treatments of the unborn. For either notion, a version of the first welfare theorem holds.

notion of Pareto-efficiency when fertility is exogenous. We then discuss how these concepts are related to each other. We also give results regarding the existence of efficient allocations and derive planning problems that partially characterize the set of efficient allocations. We prove a version of the first welfare theorem for each of them.

To do this, we provide a fairly general, general equilibrium formulation of fertility choice. Naturally, such a formulation will be embedded in an overlapping generations framework. Each decision maker has a fixed set of potential children and decides how many of them will be born. Models of fertility also naturally involve external effects across agents in the economy. We allow for any individual's utility to depend on the consumption of other family or dynasty members. This includes the Barro and Becker (1989) formulation of fertility along with many others. In addition to this utility externality, there is another more subtle one. From the point of view of the potential children, this is a model in which their choice set is dependent on the actions of other agents in the economy. If the parent chooses that they will not be born, they have effectively no choices.

As is usual in models with external effects, there is no presumption that individual behavior will aggregate to an efficient outcome. However, in models of fertility, it is commonly assumed that mechanisms exist for transfers inside the family. Following this logic, we divide the efficiency question into two pieces: efficient transfer systems within a dynastic family and efficient trade across dynasties. First, we show using standard arguments that if all trade across dynasties is done at common, parametric prices and there are no external effects across families, equilibrium is efficient as long as the dynasty problem is solved efficiently internally. Second, we give sufficient conditions for a noncooperative implementation of the dynastic game to be efficient. We discuss two extreme cases that guarantee efficiency of the family game. In the first case, dynasties are perfectly altruistic, which eliminates the potential time consistency problem among family members and thereby assures efficiency. This includes the Barro-Becker model as a special case. In the second case, if contracts between parents and children are rich enough, so that parents can effectively dictate their children's actions, then efficiency is also guaranteed, irrespective of the preference details. Other games and preference specifications may lead to equilibrium inefficiencies.

Our approach allows us to easily distinguish between two potential reasons for concern about overpopulation that have been at the center of the more recent debates on population. The first of these is the existence of scarce factors and the 'crowding' of these factors that results when the population is 'large.' The second is the potential increase in pollution (e.g., emission of greenhouse gases) as population grows. We show that scarce factors do not, in and of themselves, give rise to inefficiencies in population. Rather, this externality is 'pecuniary' with effects manifested in price changes.[2] In contrast, if true external effects exist that are related to population size, not surprisingly, individual choices do not necessarily lead to efficient population sizes. This is true both when the external effects are negative, like pollution, and when they are positive, e.g. knowledge spillovers (Romer 1987) or human capital externalities (Lucas 1988).[3] Of course, part of the debate about overpopulation is a question of distribution of resources, i.e. which of many efficient allocations is the best one. While our concepts have nothing to say about optimal redistribution among agents, we believe that identifying inefficiencies is an important first step towards such an even more ambitious goal.

The problem that Pareto efficiency is not well-defined in the endogenous population context has been long recognized in the literature. The debate over alternative concepts dates back to at least Mill (1965) and Bentham

[2]This is similar to the arguments made in Willis (1987) and Lee and Miller (1990).

[3]Interestingly, Keynes was one of the first authors to argue that population growth was too low in England in the 1920s and that this was a cause for a reduction in inventive activity and hence stagnation. (See Zimmermann 1989)

(1948) who propose per-capita utility and the sum of utilities, respectively, as alternative welfare concepts.[4] Early papers employ these alternative social welfare functions in the context of models where children do not affect preferences and parents do not choose fertility (e.g. Samuelson 1975 and Dasgupta 1969). The more recent literature assumes that a parent's utility depends on consumption, utility, and/or number of children, and uses the Millian and Benthamite criterion to compare population sizes in equilibrium with the optimal one (Nerlove, Razin, and Sadka (1987, 1989) and Razin and Sadka (1995)). Eckstein and Wolpin (1985) maximize utility of a representative agent instead. Such criteria, however, typically give one optimal allocation and are very different in spirit from an efficiency concept that usually contains a large number of allocations.

A small recent literature addresses the question of optimal populations using a Paretian approach. Schweizer (1996) and Conde-Ruiz et al. (2004) are most closely related to our approach. Each paper proposes a new efficiency concept and proves versions of the first and second welfare theorems. However, these papers propose concepts that are sufficiently less general than ours, defined only for symmetric environments and they focus exclusively on allocations that are identical for all people within a generation. Michel and Wigniolle (2003) use a concept that compares utilities generation by generation. Within the context of a specific model they give an example that shows that the concept of Golden Rule should be modified in the context of endogenous populations. Willis (1987) also attempts to analyze whether general equilibrium models with endogenous fertility lead to Pareto-efficient allocations. Willis does this, however, without formally defining Pareto-efficiency for these environments. Instead, Willis studies the solution to a planning problem and shows under what conditions it coincides with a competitive equilibrium.

An alternative approach is that from the Social Choice Literature. There,

[4]See Zimmermann (1989) for an excellent summary of the historic debate.

authors use an axiomatic approach to derive representation theorems for social orderings which include population size as one of the choices (see for example Blackorby, Bossert, and Donaldson (1995), Broome (2003, 2004)).[5] These representation theorems have a particularly simple and intuitive form known as critical level utilitarianism – a new person should always be added to the population as long as the value to society of doing this exceeds some critical level. As with the Millian and Benthamite criterion, the goal of this literature is to determine one optimal population size. Our approach is different and complementary in that it gives definitions that are analogous to the usual Pareto Frontier. As is typically the case even without the issues of endogenous fertility, this gives a large set of efficient outcomes while the social choice approach typically gives only one (for each critical level). On the plus side, our approach requires only ordinal comparisons and hence, no judgements about the meaning of interpersonal comparisons of utility, or issues about 'scaling' of utility functions is necessary. In addition, our approach naturally lends itself to addressing questions concerning the efficiency of privately chosen fertility levels without adding in the extra issues inherent to distributional questions.

Finally, a few authors have pointed out various reasons for why the private and social costs of having children could differ (Friedman (1972), Chomitz and Birdsall (1991), Lee and Miller (1991), Simon (1992), and Starrett (1993)). These papers informally discuss types of externalities that could arise in the context of fertility choice, but none provides a formal concept or the tools to thoroughly address the efficiency question.

The remainder of our paper is organized as follows. In Section 2, we introduce notation. In Section 3, we give definitions of our two notions of Pareto-optimality, give some simple examples and discuss some properties of the concepts. Section 4 contains the development of the analog of the first welfare theorem for settings in which population is endogenous and

[5] Section 6 of Blackorby, Bossert and Donaldson (2002) provides an excellent survey.

the decision-making unit is a family. In Section 5, we show that the Barro and Becker (1989) model of fertility choice is one example of a model in which our form of dynastic maximization holds and hence, the population chosen in equilibrium is efficient. Section 6 is devoted to discussing various applications of the concepts. In it we show that the theory allows us to make a tight distinction between two possible sources of inefficiency – scarcity of resources and global external effects. We show that resource scarcity per se (e.g., land crowding) does not give rise to inefficient fertility, while the presence of global external effects does. Finally, we present an example of what might cause family maximization to fail. Section 7 concludes.

2 Notation and Feasible Allocations

Dasgupta (1995, p. 1899) points out that "developing the welfare economics of population policies has proved to be extremely difficult: our ethical intuition at best extends to actual and future people, we do not yet possess a good moral vocabulary for including potential people in the calculus." In this section, we aim to make progress on this dimension by providing a new framework that makes extending the tools of modern welfare economics to questions of optimal populations possible. An important component of our framework is an explicit dynastic structure, something that has been largely ignored in the literature. The advantage of an explicit dynastic structure is threefold. First, it allows for external effects (e.g. altruism) between family members. It follows that even if the planner puts zero weight on a person, it might still be optimal for that person to be born, because a parent wants the child. That is, in our framework we take people's preferences about other people explicitly into account. Second, we make it explicit that creating another person is costly, and that this cost might not always be transferable (e.g. the time cost of a mother nursing a baby). Thirdly, it introduces a natural asymmetry between people who are alive for sure (the initial generation) and those that might or might not be born (everyone else).

Consider an overlapping generations economy, where each generation makes decisions about fertility. For simplicity, each agent is assumed to live only for one period. The initial population in period 0 is denoted by $\mathcal{P}_0 = \{1, ..., N\}$. Each person can give birth to a maximum of $\bar{f}$ children.[6] For each period t, the potential population, $\mathcal{P}_t$, is defined recursively as $\mathcal{P}_t \equiv \mathcal{P}_{t-1} \times \mathcal{F}$, where $\mathcal{F} = \{1, ..., \bar{f}\}$, and we denote by $\mathcal{P}$ the population of all agents potentially alive at all dates. Simply put, $\mathcal{P}$ is the set of all individuals that might be, depending on fertility choices, nodes of one of the N family trees, one for each time 0 agent. Then, an individual born in period t is indexed by $i^t \in \mathcal{P}_t$ and can be written as $i^t = (i^{t-1}, i_t)$, specifying that i^t is the i_tth child of the parent i^{t-1}. For example, $i_t = (1, 3, 2)$ means that person i_t is the second child of the third child of person $1 \in \mathcal{P}_0$. We often simply write i because the length of the vector already indicates the period in which the agent was born. Similarly, a fertility plan, denoted by f, is a description of the number of children born to each agent. Thus, $0 \leq f(i) \leq \bar{f}$ for all $i \in \mathcal{P}$. Each fertility plan f implicitly defines the subset (of $\mathcal{P}$) of individuals actually born under the plan f. This set will be denoted by $I(f)$ and is defined recursively by first, $i_0 \in I(f)$ for $i_0 \in \mathcal{P}_0$; for $i_0 \in \mathcal{P}_0$, $(i_0, i_1) \in I(f)$ if and only if $i_1 \leq f(i_0)$, etc. Let $I_t(f) = I(f) \cap \mathcal{P}_t$ denote the set of people alive in period t under the fertility plan f. $I(f)$ is the set of N actual family trees realized under the fertility plan f, one for each time 0 agent, or dynasty head. For $i_0 \in \mathcal{P}_0$, let D_{i_0} be the set of potential descendants of i_0 including i_0 himself. That is $i = (\hat{\imath}, i_1, \ldots, i_t) \in D_{i_0} \Leftrightarrow \hat{\imath} = i_0$. Note that $D_i \cap D_{i'} = \emptyset$ if $i \neq i'$. We will call D_i 'dynasty i.' Then, we can write $f = (f_{i_0})_{i_0 \in \mathcal{P}_0}$ when it

[6]Throughout most of the paper, we will assume that the number of children possible is discrete. Many of the models of fertility choice (e.g., Barro and Becker (1989)) allow for non-integer choices. Much of the the analysis presented here can be done in this framework as well (see Golosov et al (2006)). Finally, note that we assume that individuals have children, not couples. This is done to simplify the development that follows.

is necessary to distinguish between the fertility plans for different dynasties. For any fertility plan f, we will use the notation $I(f_i)$, $i \in \mathcal{P}_0$, to denote i and all of $i's$ descendants under the plan – $I(f_i) = I(f) \cap D_i$. Note that $I(f_i)$ does not depend on the f_{-i}, but only on f_i. We denote the set of all fertility plans by F.[7] Figure 1 illustrates the notation graphically in a 2-period setting.

We assume that there are k goods available in each period. Goods will be interpreted in a broad sense here – included are labor, leisure, capital services, etc. Given any fertility plan, a consumption plan, x, is a determination of the level of consumption of these k goods for each person that is actually born. That is, $x : I(f) \to \mathbb{R}^k$, where $x(i) \in \mathbb{R}^k$, represents the consumption of agent $i \in I(f)$. There is one representative firm, which behaves competitively. The technology is characterized by a production set, $Y \subset \mathbb{R}^{k\infty}$, that describes all feasible input-output combinations. An element of the production set is denoted by $y \in Y$. We will write $y = \{y_t\}_{t=0}^{\infty}$, where $y_t = (y_t^1, \ldots, y_t^k)$ is the projection of the production plan onto time t.

An allocation is then given by a fertility plan, a consumption plan and a production plan – (f, x, y). We will denote by A the set of all allocations, and, for $i \in \mathcal{P}$, we will use $A(i)$ to denote the set of all allocations in which i is born. When it is important to distinguish the choices individual i makes from those made by the other agents, we will use the notation $(f(i), x(i); f(-i), x(-i))$.

We assume that each potential agent is described by both an endowment of goods and preferences. We will use $e(i) \in \mathbb{R}^k$ to denote individual i's endowment and note that $e(i)$ will be irrelevant in all that follows if $i \notin I(f)$. To simplify, we assume that preferences are described by a utility function, denoted by $u_i(f, x)$, which we allow to depend on the entire fertility and consumption plan components of the allocation. We do this to allow for the

[7]Formally, $f : \mathcal{P} \to \{0, 1, ..., \bar{f}\}$. We only consider 'feasible' fertility plans – those for which $f(i^t) = 0 \Longrightarrow f(i^t, i) = 0$ for all $i \in \mathcal{F}$. F is then the set of these feasible fertility plans.

possibility of external effects across members of a family. For example, this specification allows utility to depend on fertility choices and the consumption of one's children etc. Below we will add an assumption restricting utility to depend only on fertility and allocations within one's *own* dynasty.

We consider two possible assumptions for the domain of u_i:

Assumption 1 $\mathcal{P}$ *for each* $i \in \mathcal{P}$*, there is a well defined, real-valued utility function* $u_i : A \to \mathbb{R}$*.*

Assumption 2 $\mathcal{A}$ *for each* $i \in \mathcal{P}$*, there is a well defined, real-valued utility function* $u_i : A(i) \to \mathbb{R}$*.*

The difference between these two assumptions is that in the first, we assume that utility is well defined for all potential agents, even for plans in which they are not born. In the second, we assume that utility is only defined for an individual over those allocations in which he is born. We will use these different notions in our definitions of efficiency that follow below.

There is a long-standing debate in the moral philosophy literature on what the utility of unborn people should be (see for example Singer (1993)). When considering preferences about adding new people to the status quo, there are three ways of thinking about this: (i) What are the preferences of the parents, siblings, and anyone else who feels potentially altruistic towards the newborn? (ii) How does the newly added person feel about this? (iii) What are the preferences of "society as a whole." Parental preferences (i) is probably the least controversial concept and most models of endogenous fertility include some sort of altruistic preferences like this – either from parents to children, from children to parents, or both. This implies that there is a trade-off between having a child and not. Such preferences can also easily be derived from observed choices.[8] Other approaches to efficient fertility choice (like the social welfare approach of Blackorby, Bossert, and Donaldson

[8]See Dasgupta (1994) for an ethical discussion of how parents *should* value fertility.

(1995)) make explicit assumptions about societal preferences (i.e., iii) while we do not.[9] Finally, do people have preferences about being born or not? And if so, what are these preferences? These are hard questions. Although we will sometimes assume that these preferences are well defined (i.e., Assumption 1 holds), we will only use this assumption for one of our concepts of efficiency. For the second, which we call $\mathcal{A}$-efficiency below, we will only use assumption 2. Thus, in $\mathcal{A}$-efficiency, the value of an additional child is based exclusively on the extra utility brought about to parents, grandparents, siblings, etc. Our second concept, $\mathcal{P}$-efficiency, does require well-defined preferences that include the state in which an individual is not born. However, the results that we will prove (equilibrium fertility choice is $\mathcal{P}$-efficient) do not require assumptions on the form of these preferences – only that they exist.

Each individual that is born has a set of fertility and consumption plans that is feasible for them. For simplicity, we will assume that this is the same for everyone and will denote it by $Z \subset \{0, 1.., \bar{f}\} \times \mathbb{R}^k$. The simplest version of this would have $Z = \{0, 1.., \bar{f}\} \times \mathbb{R}^k_+$ so that any choice of fertility level and any non-negative consumption is allowed. Since some models of fertility put restrictions on the joint choices of consumption and fertility (e.g., parents must care for their own children), we allow for the extra generality in Z. Most models of fertility also have a transferable cost of child production. Let $c(n) \in \mathbb{R}^k_+$ be the goods cost of having n children. We assume that this is the same for everyone for simplicity.

Assumption 3 $c(0) = 0$, *and* $c(n)$ *is strictly increasing in* n.

We can now define feasibility for this environment.

Definition 1 *An allocation* (f, x, y) *is feasible if*

1. $(f(i), x(i)) \in Z$, *for all* $i \in I(f)$,

[9] See also section 3.3 for an explicit comparison of our approach with theirs.

2. $\sum_{i \in I_t(f)} x(i) + \sum_{i \in I_t(f)} c(f(i)) = \sum_{i \in I_t(f)} e(i) + y_t$ *for all* t,

3. $y \in Y$.

3 Efficient Allocations

The formulation above turns models with an endogenous set of agents into one with a fixed set of potential agents, but with external effects in preferences, restrictions on what those potential agents that are not born can choose and, possibly, domain restrictions on their utility functions. An advantage of this construction is that we can use, as a first cut, the normal notion of Pareto-efficiency if utility functions are defined everywhere (i.e., if Assumption 1 is satisfied). We call this concept $\mathcal{P}$-efficiency, where $\mathcal{P}$ refers to populations. This concept treats born and unborn people symmetrically and preserves the principle of 'inclusiveness' of the usual Pareto criterion when comparing two allocations – every potential agent is 'consulted' and one allocation dominates if and only if it is at least as good for all agents.

If utility functions are not defined for unborn agents over allocations in which they are not born (i.e., only Assumption 2 is satisfied) it is not possible to adopt such a strong notion of inclusiveness in the Pareto criterion. Indeed, if one goes to the opposite extreme and assumes that it is not possible to assign utilities to the unborn agents for any allocation in which they are not born, it is only possible, when comparing two allocations, to compare the utilities of agents that are alive in both. Our second notion of efficiency uses this reasoning exactly – when comparing two allocations, (f, x, y) and (f', x', y') we compare the utilities of *all* agents that are alive in both, $I(f) \cap I(f')$. We call this second version $\mathcal{A}$-efficiency, since it focuses on *alive* agents. It is important to note that this does not mean that the consumption, etc., of a potential child is not considered, rather that these enter only through the utility of other, alive, agents through familial external effects (e.g., parental altruism, etc.)

As we will see later, many of our results hold for both definitions of efficiency, but we will also see that in specific applications the choice of concept matters.

3.1 Basic Concepts

$\mathcal{P}$-efficiency does not distinguish between agents who are born and not born in its treatment beyond what is implicit in feasibility and preferences. It is defined as follows.

Definition 2 *A feasible allocation* (f, x, y) *is* $\mathcal{P}$*-efficient if there is no other feasible allocation* $(\hat{f}, \hat{x}, \hat{y})$ *such that*

1. $u_i(\hat{f}, \hat{x}) \geq u_i(f, x)$ *for all* $i \in \mathcal{P}$,

2. $u_i(\hat{f}, \hat{x}) > u_i(f, x)$ *for at least one* $i \in \mathcal{P}$.

Let $\mathbb{P}$ denote the set of all $\mathcal{P}$-efficient allocations. If for any allocation (f, x, y) there exists some feasible allocation $(\hat{f}, \hat{x}, \hat{y})$ such that (1) and (2) in the definition above are satisfied, then we say that $(\hat{f}, \hat{x}, \hat{y})$ $\mathcal{P}$*-dominates* (f, x, y). It follows that under Assumption 1, $\mathcal{P}$*-domination* is a well defined ordering of the feasible set. It is not complete (typically), but it is transitive and irreflexive. These are all properties of the usual notion of Pareto Optimality in settings with fixed populations as well.

This definition seems to be the most natural extension of Pareto efficiency in the framework with endogenous fertility. It has, however, two important deficiencies. First, to choose which allocations are efficient it is necessary that the preferences of the unborn agents are well defined – Assumption 1 must be satisfied. Unlike alive agents, whose preferences could be at least deduced from their observed choices, preferences of the unborn agents are inherently impossible to observe.[10] Therefore, the set of efficient allocations will depend on an arbitrary choice of the preferences for the unborn. This leads to a second deficiency. One natural benchmark level of utility for the unborn is that being alive is always preferred. We can formalize this assumption in the following way:

Assumption 4 *a) For all $i \in \mathcal{P}$, there exists $\bar{u}_i$ such that for all (f, x), if $i \in \mathcal{P} \backslash I(f)$, then $u_i(f, x) = \bar{u}_i$,*

b) For all (f, x) and all i, if $i \in I(f)$, then $u_i(f, x) > \bar{u}_i$.

Note that if Assumption 4 is satisfied with $\bar{u}_i = 0$, then $\mathcal{P}$-efficiency satisfies what Dasgupta (1994) calls the *Pareto-Plus Principle*: An allocation with an additional person enjoying a positive utility level is preferred to an allocation without the additional person but otherwise identical.

It is easy to see that under this assumption it is impossible to have a population level which is too high. Any allocation with fewer agents will necessarily decrease the utility of the agents who were born under original allocations, and therefore the new allocation cannot be more efficient in a $\mathcal{P}$-sense.

Our second notion of efficiency overcomes these difficulties by treating born and unborn potential people asymmetrically. In this way, efficient allocations *do not* depend on preferences of the unborn, or even whether such preferences are defined at all – that is, only Assumption 2 needs to be satisified (but it is also defined if Assumption 1 is satisfied).

Definition 3 *A feasible allocation (f, x, y) is $\mathcal{A}$-efficient if there is no other feasible allocation $(\hat{f}, \hat{x}, \hat{y})$ such that*

1. $u_i(\hat{f}, \hat{x}) \geq u_i(f, x) \quad \forall i \in I(f) \cap I(\hat{f})$,

[10] Note, however, that preferences of people that are not yet born can also not be deduced from observed choices. Yet it is a standard assumption made in overlapping generations models that utility functions for all (future) generations are well-defined.

2. $u_i(\hat{f}, \hat{x}) > u_i(f, x)$ *for some* $i \in I(f) \cap I(\hat{f})$.

The definitions of the set of $\mathcal{A}$-efficient allocations, $\mathbb{A}$, and $\mathcal{A}$-dominating allocations are defined analogously to $\mathcal{P}$-efficiency.

This definition differs from $\mathcal{P}$-efficiency in that only a subset of the potential population is considered when making utility comparisons across allocations. An allocation is superior if no one who is alive in both allocation is worse off and at least one person alive under both allocations is strictly better off. Since utility comparisons are made only for the agents who are in fact born, (i.e., $i \in I(f) \cap I(\hat{f})$) it has the added advantage of not requiring utility functions to be defined for agents who are not born. We call it $\mathcal{A}$-*efficiency* because only 'alive' agents are considered. (Note that even agents that are not born count in $\mathcal{A}$-*efficiency,* at least indirectly, since they enter the utility functions of their parents, etc.) It has the disadvantage that the set of agents considered in welfare comparisons depends on the two allocations being considered. This can, in some cases, cause cycles and hence, non-existence.[11] However, we show in Section 3.4 that generically (in utility functions) the set of $\mathcal{A}$-efficient allocations is non-empty.[12]

The notions $\mathcal{P}$ and $\mathcal{A}$ efficiency extend the standard notion of Pareto efficiency. In particular, given any feasible allocation (f^*, x^*, y^*), we can consider the standard Pareto ranking over allocations holding fixed the population at $I(f^*)$. The next proposition shows that if (f^*, x^*, y^*) is $\mathcal{P}$-efficient (resp. $\mathcal{A}$-efficient) (x^*, y^*) is a Pareto efficient allocation in the usual sense.

Proposition 1 *a) If Assumption 4a holds and if* (f^*, x^*, y^*) *is* $\mathcal{P}$*-efficient, then, the consumption/production plan* (x^*, y^*) *is an allocation that is Pareto-optimal among the agents in* $I(f^*)$.

[11] Note that $\mathcal{A}$-domination need not be transitive.

[12] Conde-Ruiz et al (2004) propose a modification of $\mathcal{A}$-efficiency that requires symmetry among all people born in the same period. This modified concept guarantees existence, but is substantially less general as it does not allow for heterogeneity in preferences, endowments, or allocations at a point in time.

b) If (f^*, x^*, y^*) is $\mathcal{A}$-efficient, the consumption/production plan (x^*, y^*) is an allocation that is Pareto-optimal among the agents in $I(f^*)$.

Proof. Let (f^*, x^*, y^*) be a $\mathcal{P}$- efficient ($\mathcal{A}$-efficient) allocation. Suppose that there is some allocation $(\tilde{x}, \tilde{y})$ that is feasible given the set of alive people $I(f^*)$[13] that dominates (x^*, y^*) in the usual Pareto sense. It is immediate that in this case, $(f^*, \tilde{x}, \tilde{y})$ necessarily $\mathcal{A}$-dominates (f^*, x^*, y^*). That $(f^*, \tilde{x}, \tilde{y})$ also $\mathcal{P}$-dominates (f^*, x^*, y^*) follows from Assumption 4a. Therefore (f^*, x^*, y^*) could not be $\mathcal{P}$-efficient ($\mathcal{A}$-efficient).

The converse of this proposition will not necessarily hold even if Assumption 4 holds. That is, even if an allocation is Pareto Efficient in the usual sense holding the population fixed, it need not be either $\mathcal{P}$- efficient or $\mathcal{A}$-efficient since welfare might be increased by changing the set of people alive.[14]

3.2 Examples

To illustrate our two notions of efficiency, we now consider two simple examples motivated by Barro and Becker (1988, 1989).

Example 1: Consider a 2-period example with only one parent, $\mathcal{P}_0 = \{1\}$. In period 0, there are e_0 units of a good that can be used either for consumption or for raising children. The cost of each child is $\theta > 0$. Parents care about own consumption and are altruistic towards each child as well. The utility function of the parent is

$$u_1(c(1), f(1); c(1,1), ..., c(1, f(1))) = \begin{cases} u(c(1)) + \beta \frac{1}{f(1)^\eta} \sum_{j=1}^{f(1)} u(c(1,j)), \text{ if } f(1) > 0 \\ u(c(1)), \text{ if } f(1) = 0 \end{cases}$$

where u is non-negative, strictly increasing and strictly concave, $0 < \beta < 1$

[13]Feasibility given a set of people is defined in the usual way.

[14]Of course if it is physically not feasible to change the set of people, then all three concepts coincide.

and $0 < \eta < 1$. The utility function of the $i-th$ potential child is given by

$$u_{(1,i)}(c(1), f(1); c(1,1), ..., c(1, f(1)) = \begin{cases} u(c(1,i)) \text{ if } 1 \leq i \leq f(1) \text{ (i is born)} , \\ \bar{u} \text{ if } f(1) < i \text{ (i is not born).} \end{cases}$$

In the example, we assume that Assumption 1 holds: utility of the child is well-defined when not born. Note that without this assumption $\mathcal{P}$-efficiency is not defined, but, $\mathcal{A}$-efficiency is unchanged. Further, we assume that each child, if born, has an endowment of the consumption good $e(1,i) = e_1 > 0$. To simplify, we assume that e_1 is not transferable.[15] Then the possible utility levels for the parent are given by

$$\begin{aligned} W(f(1)) &= u(e_0 - \theta f(1)) + \beta \frac{1}{f^{\eta}(1)} \sum_{1 \leq j \leq f(1)} u(e_1) \\ &= u(e_0 - \theta f(1)) + \beta f(1)^{1-\eta} u(e_1), \end{aligned}$$

where $f(1) \in \{0, 1, \ldots, \bar{f}\}$. We assume that $W(f(1))$ has a unique maximum, f^*, with $0 < f^* < \bar{f}$. Further, let $e_0 > \theta \bar{f}$.

First, consider the case where $u(e_1) > \bar{u}$. In this case, it is straightforward that no fertility level less than f^* is efficient (both $\mathcal{A}$ or $\mathcal{P}$): increasing fertility to f^* from such a level *strictly* increases the utility of the parent and the added children and does not lower the utility level of anyone. It also follows that any $f \in \{f^*, ..., \bar{f}\}$ along with $c(1) = e_0 - \theta f$ gives a $\mathcal{P}$-efficient allocation. This is because, any increase in fertility would necessarily lower the utility of the parent, and any decrease would lower the utility of the children that are no longer born.

In contrast, f^* is the unique $\mathcal{A}$-efficient fertility level because any fertility level higher than f^* is $\mathcal{A}$-dominated by f^*: moving to f^* strictly increases the utility of the parent and does not change the utility of the children that are still born.

[15] We assume that each born period 1 child must consume her own endowment. Adding the possibility of redistributing the endowments of period 1 children increases the size of the sets of efficient outcomes in the usual way.

If instead $u(e_1) < \bar{u}$, the set of $\mathcal{P}$-efficient allocations correspond to all fertility levels in the set $\{0, ..., f^*\}$, while the unique $\mathcal{A}$-efficient allocation still has $f = f^*$ as above.

In this example, the set of $\mathcal{P}$-efficient allocations is much larger than the set of $\mathcal{A}$-efficient allocations, a difference that holds more generally, as we will discuss below. The example shows that larger populations can be $\mathcal{A}$-dominated by smaller ones if reducing the size of the population does not lower the utility of those agents that are still born. Thus, $\mathcal{A}$-efficiency does not suffer from the difficulty pointed out above for $\mathcal{P}$-efficiency.

Example 2: One might get the impression from Example 1 that $\mathcal{A}$-efficiency corresponds to maximizing the utility of the dynasty head. This is not true in general, however. Consider a slightly modified version of Example 1 in which goods from period 0 can be stored, with no loss, to period 1 and goods can be transferred among the period 1 agents. Feasibility here is captured in the two constraints:

$$c(1) + f(1)\theta + \sum_{j=1}^{f(1)} c(1,j) \leq e_0 + f(1)e_1 \text{ and } c(1) \leq e_0 - f(1)\theta$$

Again first consider the unique outcome that is best for the dynasty head: For simplicity, assume that $f^* = 1$ and $c^*(1) > 0$. Then $c^*(1,1) \geq e_1$ follows from feasibility. This allocation is clearly $\mathcal{A}$-efficient. However, this is not the only $\mathcal{A}$-efficient allocation. Lowering consumption of the parent by δ and increasing the consumption of the child by the same amount will also lead to an $\mathcal{A}$-efficient allocation as long as $u(c^*(1) - \delta) + \beta u(c^*(1,1) + \delta) > u(e_0)$.[16] The logic is the same as with regular Pareto efficiency: there are two agents who disagree about the distribution of resources, and efficiency has nothing to say about redistribution, hence, many allocations are efficient.

So far, one could still suspect that *fertility* in any $\mathcal{A}$-efficient allocation is always equal to the most preferred choice of the dynasty head. However,

[16] If δ is such that the condition is violated, then the parent strictly prefer zero children and hence the allocation is not $\mathcal{A}$-efficient.

this is not true either. If $f^* > 1$, then there are also typically $\mathcal{A}$-efficient allocations with $f < f^*$. To see this, let $e_0 = 100, e_1 = 0, \theta = 24, \beta = 1, \eta = 0$ and $u(c) = \sqrt{c}$. For these parameters, the parent's most preferred allocation is to have two children and split resources evenly, i.e. $c(1) = c(1,1) = c(1,2) = \frac{100-48}{3}$, which gives utility 12.48 to the parent, and is $\mathcal{A}$-efficient. Now consider the allocation that maximizes the parent's utility *conditional* on having only one child: $\hat{c}(1) = \hat{c}(1,1) = \frac{100-24}{2} = 38$. Clearly, this allocation is strictly preferred by the child, and worse for the parent, whose utility under this allocation is only 12.33. To see that this allocation is also $\mathcal{A}$-efficient note that it cannot be $\mathcal{A}$-dominated by the allocation with zero children, as this would give only utility $\sqrt{100} = 10$ to the parent. It also cannot be $\mathcal{A}$-dominated by any allocation with two children, as any such allocation would have to give at least 38 to the first child, which leaves only $\frac{100-38-48}{2} = 7$ each for the parent and the second child, and parental utility decreases to 11.46.

There are also other types of examples where $\mathcal{A}$-efficiency differs from dynastic head maximization. These include examples where children prefer to be in families with a large number of children (so that fertility levels higher than f^* are $\mathcal{A}$-efficient) and examples where parents and children do not have the same utility functions over the consumption of the child (i.e., there is a time consistency problem within the dynasty – altruism is imperfect). To save on space, we don't include any examples of this sort here.

3.3 Comparisons to the Social Choice Literature

An alternative approach to optimal population appears in the Social Choice Literature (see Blackorby, Bossert, and Donaldson (1995, 1997, 1999)). The main contribution of this literature are characterization theorems of the functional form of Societal Welfare Functions (SWF) under a variety of alternative specifications of axioms.[17] Blackorby, Bossert, and Donaldson (1995) – henceforth BBD – deals explicitly with the case of variable populations and is thus the most relevant for comparison with our approach. In that paper, they derive conditions under which the SWF is of the "critical level generalized utilitarianism" form. That is, the value to society of an allocation (f, x) is given by[18]

$$W(f, x; \alpha) = \sum_{i \in I(f)} [g(u_i(f, x)) - g(\alpha)].^{19} \quad (1)$$

The special case where $g(z) = z$, is known as critical level utilitarianism and is the case we will focus on here. Note also that critical level utilitarianism reduces to the Benthamite welfare function when $\alpha = 0$. For social preferences of these forms, (f, x) is socially weakly preferred to $(\hat{f}, \hat{x})$ if and only if $W(f, x; \alpha) \geq W(\hat{f}, \hat{x}; \alpha)$. Here, BBD describe α as a "societal preference parameter," with the interpretation that a new person contributes to social welfare only if his utility is at least α. Thus, under this approach, the optimal population (or time series of populations) is that (f, x) that maximizes $W(f, x; \alpha)$ subject to feasibility. If there are no direct utility connections between potential people and resources are not transferable, it follows that the rule for finding the optimal population is to keep adding more people until it is no longer possible to add a person and give her utility level α.

One insight from this literature is that if there is no altruism, and if $\alpha \leq 0$, then the "repugnant conclusion" follows: It is always optimal to have the maximal feasible number of people alive. This has been generally considered a non-desirable property, and hence an insight of BBD is that critical-level utilitarianism with $\alpha > 0$ avoids the repugnant conclusion. An immediate conclusion then is that life between neutrality (defined as 0

[17]Axioms used in this literature that are violated by our approach are continuity, completeness, and, in the case of $\mathcal{A}$-efficiency, transitivity.

[18]We abstract from the possibility of production here to simplify the presentation.

[19]BBD assume that the maximal number of people is finite, which guarantees that this sum is well-defined.

utility) and α should be prevented. As pointed out by Hammond (1988), an alternative method for ruling out the repugnant conclusion is to assume that parents care about the well-being of their children.[20] As is also evident from this representation, the BBD axioms imply that there cannot be social discounting of utility between lifetimes starting at different calendar dates although discounting of consumption at different dates by a given individual is allowed.

Our notions of $\mathcal{P}$ and $\mathcal{A}$-efficiency on the other hand, are designed to address the question of whether or not equilibria in models with endogenous fertility are efficient. Therefore we want a set-up where people are connected (who is the child of whom?) and where there is some altruism or other benefit from children – without this, the equilibrium would be trivial: no children are born and the world ends after period 0. It follows that Hammond's comment applies to our set-up, which means the concerns raised by BBD are not relevant for us. Similarly, the result of no social discounting applies only in so far that "society" should not discount future generations. However, there might of course be discounting through the parent's preferences, even in the BBD setting. Also, note that BBD require that the allocation where no one is alive is always possible, whereas we start with an initial generation, so that $\mathcal{P}_0 \subset I(f)$ for any feasible f.

Comparison with P and A-efficiency

It is important to note that Pareto optimality is inherently a very different concept from social welfare maximization. Typically the set of Pareto optima is very large and is impicitly agnostic about alternative welfare distributions across agents. On the other hand, the SWF-maximizer (with some assumptions) is unique and does make judgements about alternative distributional arrangements. Moreover, social welfare functions assume interpersonal comparability of utility, and because of this are cardinal, not ordinal, which is

[20]BBD are fairly explicit that $u_i(f, x)$ is not supposed to capture overall preferences, but only a measure of individual well-being.

not required for notions of efficiency.[21]

The simplest way to compare BBD optimal allocations and those that are $\mathcal{P}$- or $\mathcal{A}$-efficient is to examine the three concepts in the context of a simple example. For this, we use Example 1 outlined above, and assume that $g(z) = z$, i.e., we restrict attention to the critical level utilitarianism case. Recall that in that example, assuming that $u_{(1,j)}(e_1) > \bar{u}$, the set of $\mathcal{P}$-efficient fertility levels is given by $\mathbb{P}_f = \{f^*, ..., \bar{f}\}$, while $\mathbb{A}_f = \{f^*\}$ – recall that everyone consumes their own endowment in Example 1. Let $S(\alpha)$ be the set of maximizers of $W(f, x; \alpha)$ for a given α. Then, comparing the three concepts gives the following results:

i) $S(\alpha)$ is decreasing in α;

ii) $S(\alpha^*) = \mathbb{A}_f$ for $\alpha^* = u(e_1)$.

iii) For all $\alpha \leq \alpha^*$, $S(\alpha) \subset \mathbb{P}_f$.

iv) For $\alpha > \alpha^*$, if $f \in S(\alpha)$, $f < f^*$ and so $S(\alpha) \cap \mathbb{P}_f = S(\alpha) \cap \mathbb{A}_f = \emptyset$.

Figure 2 gives a graphical summary of these results.

From this example we can see that there is no uniform, obvious relationship between $S(\alpha)$ and either $\mathbb{A}_f$ or $\mathbb{P}_f$. For low critical utility levels (α), $S(\alpha) > f^*$ but is a subset of $\mathbb{P}_f$. For high values of α the opposite is true, $S(\alpha) < f^*$.

Since there are no external effects in this example that would suggest privately chosen fertility (i.e., $f = f^*$) is too high, it would be difficult to rationalize any fertility level below that as being reasonable. Higher values for f can be rationalized, but only if one is willing to assume that the utility of being born is higher than that of not – this is what $\mathcal{P}$ does effectively.

This suggests that $\mathcal{A}$-efficiency might be useful as a way of offering some guidance in choosing α in settings like this, viz., choose α so that $S(\alpha) = \{f^*\}$. In our example, this would be $\alpha = u(e_1) = 1/\beta(f^*)^\eta u'(e_0 - \theta f^*)\theta$. Then, the critical level should be higher the higher θ, the marginal cost, and

[21] It is straightforward to check that both $\mathcal{P}$ and $\mathcal{A}$-efficiency are invariant to arbitrary, monotone transformations of utility functions of any subset of the agents.

lower if the marginal utility loss to the parent due to the extra child is low, $u'(e_0 - \theta f^*)$.

Extending the example to three periods, it is easy to show that the implied α would have to differ across generations. This seems also natural if there is for example technological progress in an economy, then what is considered an "existence minimum" in a society typically depends on the average standard of living, not some absolute amount. BBD, on the other hand, derive the same critical level α for everyone. The reason is that BBD consider all people as potential and ask how many lives should ideally exist.[22] Considering *all* people as potential motivates an anonymity axiom, which then immediately implies the same critical level α for everyone. In our work, on the other hand, we make a clear distinction between the initial generation and potential future people.

Note also that while α is critical for $S(\alpha)$, $\mathbb{P}$ depends on $\bar{u}$ only in a very limited sense. As was shown in Section 3.2, for low $\bar{u}$, $\mathbb{P}_f = \{f^*, \ldots, \bar{f}\}$ while for high $\bar{u}$, $\mathbb{P}_f = \{0, \ldots, f^*\}$. Note that it is always true that $S(\alpha) \subset \mathbb{P}_f$ if $\alpha = \bar{u}_i$ for all i. That is, the optimal population size according to critical level utilitarianism is always $\mathcal{P}$-efficient as long as one makes the same choices for the critical level for an individual, $\bar{u}_i$, and the critical level for "society", α. In other words, if one had strong ethical preferences that life below a certain level should be prevented, then this could easily be incorporated into our $\mathcal{P}$-efficiency concepts by setting the utility of not being born to this critical level.

3.4 Properties

In this subsection we briefly discuss to what extent some standard properties of Pareto efficiency carry over into our context. We start with a partial

[22]Dasgupta (1994) labels this the *Genesis Problem* and points out important differences with set-ups in which an initial set of people exists.

characterization of efficient allocations. We then discuss conditions guaranteeing that the set of $\mathcal{P}$-efficient allocations (resp. $\mathcal{A}$-efficient) is not empty. Finally, we analyze the relationship between these two notions of efficiency. Since $\mathcal{P}$-efficiency is not defined unless Assumption 1 holds it should be understood to be assumed to hold in all the results that follow (similarly, we assume, without explicitly listing it, that *at least* Assumption 2 holds whenever $\mathcal{A}$-efficiency is being discussed).

We start with a partial characterization of $\mathcal{P}$-efficient allocations.

Result 1 *Pick any welfare weights, $\{a(i)\}_{i\in\mathcal{P}}$, such that $a(i) > 0, \quad \forall i \in \mathcal{P}$. Suppose (f^*, x^*, y^*) is a solution to the following problem:*

$$\max_{(f,x,y)} \sum_{i\in\mathcal{P}} a(i)u_i(f,x) \quad , \tag{2}$$

subject to feasibility and suppose that $\sum_{i\in\mathcal{P}} a(i)u_i(f^*, x^*) < \infty$. Then (f^*, x^*, y^*) is $\mathcal{P}$-efficient.

Proof. By way of contradiction, assume that there exists a feasible (f, x, y) that $\mathcal{P}$-dominates (f^*, x^*, y^*), where (f^*, x^*, y^*) is a solution to (2). Then $u_i(f, x) > u_i(f^*, x^*)$ for at least one i and $u_i(f, x) \geq u_i(f^*, x^*)$ for all $i \in \mathcal{P}$. Summing up, we have $\sum_{i\in\mathcal{P}} a(i)u_i(f, x) > \sum_{i\in\mathcal{P}} a(i)u_i(f^*, x^*)$, a contradiction. $\square$

In contrast to the usual characterization results, the weights $a(i)$ are required to be strictly positive for Result 1. The reason is that strict and weak Pareto-efficiency do not necessarily coincide in this context because preferences are typically not strictly monotone in all goods.[23] In other words, in environments in which weak and strong Pareto-efficiency coincide, Result 1 holds with weakly positive weights.

[23]In particular, people typically do not receive utility from consumption in periods in which they are not alive.

Result 2 *Pick any weights,* $\{a(i)\}_{i \in \mathcal{P}_0}$*, such that* $a(i) \geq 0 \;\; \forall i \in \mathcal{P}_0$*. Suppose* (f^*, x^*, y^*) *is the unique solution to the following problem:*

$$\max_{(f,x,y)} \sum_{i \in \mathcal{P}_0} a(i) u_i(f, x) \quad , \tag{3}$$

subject to feasibility and suppose that $\sum_{i \in \mathcal{P}_0} a(i) u_i(f^*, x^*) < \infty$*. Then* (f^*, x^*, y^*) *is* $\mathcal{A}$*-efficient.*

Proof. Let (f^*, x^*, y^*) be a solution to Problem (3) and assume by way of contradiction that it is $\mathcal{A}$-dominated by (f, x, y). Then there must exist a $j \in I(f^*) \cap I(f)$ such that $u_j(f, x) > u_j(f^*, x^*)$ and $u_i(f, x) \geq u_i(f^*, x^*)$ for all $i \in I(f) \cap I(f^*)$, i.e. in particular for all $i \in \mathcal{P}_0$. Note that j cannot be in $\mathcal{P}_0$ because then (f^*, x^*) would not be a maximizer of (3). But then we have $\sum_{i \in \mathcal{P}_0} a(i) u_i(f, x) = \sum_{i \in \mathcal{P}_0} a(i) u_i(f^*, x^*)$ but $(f, x, y) \neq (f^*, x^*, y^*)$, hence (f^*, x^*, y^*) is not unique, a contradiction. $\square$

That uniqueness is required in Result 2 is unusual. But, using this in conjuction with the fact that $\mathcal{P}_0 \subset I(f)$ for every feasible allocation gives the result since, any other plan must necessarily make some agent in $\mathcal{P}_0$ worse off. If the solution is not unique, and there are two with different sets of individuals born, individuals in future dates may not be indifferent between the two plans even though those in $\mathcal{P}_0$ are, and hence, the argument given may not hold. It also follows from this result that the set of $\mathcal{A}$-efficient allocations is generically nonempty, viz., if the planner's problem given here does NOT have a unique solution, utility functions can be changed by a small amount so that a unique solution is guaranteed. Then, for these perturbed utility functions, the set of $\mathcal{A}$-efficient allocations is nonempty.

From these two results, and a few technical conditions to guarantee that solutions to the problems like those given actually have solutions, it follows that both $\mathbb{P}$ and $\mathbb{A}$ are non-empty.[24]

[24] The formal proof of Result 3 is given in Golosov, Jones and Tertilt (2006).

Result 3 *Assume utility functions are continuous and uniformly bounded above and below, that* $Z \subset \{0, 1, ..., \bar{f}\} \times \mathbb{R}^k$ *is closed, that* $Y \subset \mathbb{R}^{k\infty}$ *is closed in the product topology, and that the set of feasible consumption/production plans is bounded period by period.*

a) Then the set of $\mathcal{P}$*-efficient allocations,* $\mathbb{P}$*, is nonempty.*

b) Generically, the set of $\mathcal{A}$*-efficient allocations,* $\mathbb{A}$*, is nonempty.*[25]

We turn now to the relationship between the set of $\mathcal{A}$ and $\mathcal{P}$ efficient allocations. Intuitively, one would expect that $\mathbb{A} \subseteq \mathbb{P}$ – as one need not (weakly) improve the utility of *all* agents to 'block' an allocation, hence it it typically easier to find an $\mathcal{A}$-dominating allocation than a $\mathcal{P}$-dominating allocation. However, there is a counterbalancing effect. Sometimes it may be more difficult to $\mathcal{A}$-dominate an allocation because the set of people whose utility could potentially be strictly improved is smaller. Because of this, there might exist $\mathcal{A}$-efficient allocations that are not $\mathcal{P}$ -efficient.

Example 3: Consider a two period, one good example with one parent and one potential child each of which has an endowment of $e > 0$ units of the consumption good in the period they are alive. There is a technology that allows to transfer goods between the periods with a rate 1. The cost of having a child is $\theta > 0$. The utility function of the parent is $u_1(c(1), f(1); c(1,1)) = u(c(1)) + f(1)u(c(1,1))$, and that of the potential child is $u_{(1,1)}(c(1), f(1); c(1,1)) = f(1)u(c(1,1))$. If the parameters are such that $2u(e-\theta/2) = u(e)$, then the parent is indifferent between having a child (with both consuming $c(1) = c(1,1) = e - \theta/2$) and not having one, but the child's utility is higher if born. Because of this, the allocation in which the child is born $\mathcal{P}$-dominates the one in which he is not, but it does not $\mathcal{A}$-dominate it. In this case, having the child is both $\mathcal{P}$- and $\mathcal{A}$-efficient, while

[25] Generically here means: if $\mathbb{A} = \emptyset$ for some choice of utility functions and endowments, then there is another choice of utility functions, uniformly within ε such that $\mathbb{A} \neq \emptyset$ with the same endowments.

not having the child is $\mathcal{A}$- but not $\mathcal{P}$-efficient.

Examples like this one arise due to a difference between Pareto efficiency and weak Pareto efficiency in this environment. This equivalence can break down in our context for several reasons: lack of strict monotonicity in all commodities, fertility choices are indivisible, and external effects.[26] In cases where these two notions are the same it follows that $\mathbb{A} \subset \mathbb{P}$. Even if the two notions are not the same, it is 'typically' true that 'most' of $\mathbb{A}$ is contained in $\mathbb{P}$.

To formalize this, we need some preliminary developments.

Proposition 2 *If Assumption 4 holds, if* $(f, x, y) \in \mathbb{A} \backslash \mathbb{P}$*, and if the allocation* $(\hat{f}, \hat{x}, \hat{y})$ $\mathcal{P}$*-dominates* (f, x, y)*, then:*

i) $I(f) \subset I(\hat{f})$,

ii) $u_i(\hat{f}, \hat{x}) = u_i(f, x)$ for all $i \in I(f) \cap I(\hat{f})$,

iii) $u_i(\hat{f}, \hat{x}) > u_i(f, x)$ for some $i \in I(\hat{f}) \backslash I(f)$.

Proof. Part (i) follows immediately from Assumption 4, which implies that any $\mathcal{P}$-dominating allocation always has weakly more people. Since $(f, x, y) \in \mathbb{A}$, it follows that $u_i(\hat{f}, \hat{x}) \leq u_i(f, x)$ for all $i \in I(f) \cap I(\hat{f})$. But since $(\hat{f}, \hat{x}, \hat{y})$ $\mathcal{P}$-dominates (f, x, y), it must also be true that $u_i(\hat{f}, \hat{x}) \geq u_i(f, x)$ for all $i \in \mathcal{P}$. Together this implies Part (ii). Then Part (iii) follows from (ii) together with the fact that $(\hat{f}, \hat{x}, \hat{y})$ $\mathcal{P}$-dominates (f, x, y). □

The proposition shows that the set of alive people in every $\mathcal{P}$-dominating allocation is strictly larger, and that those alive in both are strictly indifferent. If there was a way to increase the population AND increase the utility of even one of the agents in the original allocation, the allocation in question could not be $\mathcal{A}$-efficient. We will use these facts heavily in the discussion

[26] In particular, when Assumption 4 is satisfied, then preferences of unborn are locally satiated and hence, typically, weak and strong efficiency need not coincide. Thus, for these two to coincide, we would need, at a minimum, that utilities of the unborn depend on the consumption of their born relatives, even if only by a marginal amount.

that follows. Indeed, the requirement that all agents be exactly indifferent is what makes it 'rare' for an allocation to be in $\mathbb{A}\backslash\mathbb{P}$, as we shall see.

For the remainder of this section, we assume that T is finite and that there is only one good. We also assume that goods are perfectly transferable across time (both forward and backward) and that this is the only form of production that is possible.[27] Given this we can replace the production set, etc., with the following simple assumption on the aggregate feasibility constraint:

Assumption 5 . *Assume that aggregate feasibility take the form*

$$\sum_{i\in I(f)} (x(i) + c(f(i))) \leq \sum_{i\in I(f)} e(i).$$

Finally, we specialize the form of the utility functions:

Assumption 6 *Assume that the utility function of agent i in dynasty j is given by:*

$$u_i(f,x) = \begin{cases} v_i(f_j) + \sum_{i'\in I(f_j)} u_{ii'}(x(i')) & \text{if } i \in I(f) \\ \bar{u}_i & \text{if } i \notin I(f), \end{cases}$$

where $u_{ii'}$ is assumed to be strictly increasing, strictly concave and C^1, and v_i is strictly increasing in f_j.

Note that we are *not* assuming that $u_{ii'} = u_{i'i'}$, and hence, this formulation is quite general. Further note that, by construction, there are assumed to be no utility externalities across dynasties (this is an assumption we will make in more generality in the next section).

Now we are ready to state the main result regarding the relationship between $\mathcal{P}$ and $\mathcal{A}$-efficient allocations:

[27] The assumption that goods are freely transferable both foward and backward in time is a strong one. We conjecture that this is not necessary however, because in general, at efficient allocations, price-taking agents always act as if goods are freely transferable across time at the rate of exchange given by the prices that 'support' the allocation.

Proposition 3 *Assume that* $(f^*, x^*) \in \mathbb{A} \backslash \mathbb{P}$, *and that*

a) at least one $\mathcal{P}$*-dominating allocation of* (f^*, x^*), $(\hat{f}, \hat{x})$, *does not strictly increase the population of every dynasty,*

b) $x^*(i) > 0$ *for all* $i \in I(f^*)$,

c) Assumptions 4, 5 and 6 hold.

Then, there exists a sequence $\{(f_n, x_n)\}$, $(f_n, x_n) \in \mathbb{P}$ *s.t.* $(f_n, x_n) \rightarrow (f^*, x^*)$.[28]

Proof: See Appendix A.1.

The proposition shows that under relatively mild assumptions, $\mathcal{A}$-efficient allocations are either also $\mathcal{P}$-efficient, or are arbitrarily close to allocations that are.

4 Cooperation Within the Family and the First Welfare Theorem

Our economy has external effects both in utility and in consumption sets, but they are of a very limited type. By construction, the only agents in the economy who can affect i's consumption set are those that are direct predecessors of i. Moreover, in our description of the consumption sets, these agents can only affect i's choice set through their fertility decisions. In keeping with this structure, in this section we examine the validity of the first welfare theorem under the assumption that within a family (but not across families) individual agents are cooperative. That is, we formulate a notion of dynastic maximization that corresponds to a Pareto criterion within the dynasty.

We show that as long as all external effects are confined within the family, families view themselves as not affecting prices and, within the family,

[28] We will write $(f_n, x_n) \rightarrow (f^*, x^*)$ if $f_n = f^*$ for large enough n and $x_n \rightarrow x^*$ in the normal Euclidean sense.

decision making satisfies this notion of cooperation, then fertility choices are efficient. In Section 5, we address the question: Under what conditions do non-cooperative formulations of the dynastic decision problem lead to cooperative dynastic decisions in the sense required here?

Assumption 7 *If for two allocations (f, x, y) and (f', x', y'), $(f_j, x_j) = (f'_j, x'_j)$ for all $j \in D_i$ then $u_j(z) = u_j(z')$ for all $j \in D_i$.*

Assumption 8 *If j and j' are in the same dynasty, D_i, then u_j is monotone increasing in $x(j')$. That is, there are no negative external effects in consumption within the family.*

Thus, there are no external effects among agents in different dynasties and those that do exist within a dynasty are positive.

Next, we define what it means for an allocation to be optimal for a given dynasty at a given price sequence. Intuitively, an allocation is dynastically maximizing if and only if there is no way of increasing the utility of every member of the dynasty without increasing overall spending by the dynasty.

Before defining a notion of family optimization, we need to specify an ownership structure for the firm. To simplify, we will assume the firm is owned only by members of the initial generation. So let ψ_i specify the fraction of the firm that belongs to i, $i \in \mathcal{P}_0$. For a well-defined ownership structure, we need $\psi_i \geq 0$, and $\sum_{i \in \mathcal{P}_0} \psi_i = 1$.

Definition 4 *Given (p, y), a dynastic allocation for dynasty i, $(f_i, x_i) = \{f(j), x(j)\}_{j \in D_i}$ is said to be Dynastically $\mathcal{P}$-maximizing if $(f(j), x(j)) \in Z$ for all $j \in I(f_i)$ and $\sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(f_i)} (x(j) + c(f(j))) \leq \sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(f_i)} e(j) + \psi_i \sum_t p_t y_t$, and if $\nexists (\hat{f}_i, \hat{x}_i) = \{\hat{f}(j), \hat{x}(j)\}_{j \in D_i}$ such that:*

1. *$(\hat{f}(j), \hat{x}(j)) \in Z$ for all $j \in I(\hat{f}_i)$.*

2. *$u_j(\hat{f}_i, \hat{x}_i) \geq u_j(f_i, x_i)$ for all $j \in D_i$.*

3. *$u_j(\hat{f}_i, \hat{x}_i) > u_j(f_i, x_i))$ for at least one $j \in D_i$.*

4. $\sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f}_i)} (\hat{x}(j) + c(\hat{f}(j))) \leq \sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f}_i)} e(j) + \psi_i \sum_t p_t y_t$.

Dynastic $\mathcal{A}$-maximization is defined similarly.[29]

For notational simplicity in what follows, we will use Π_i to denote a dynasty's profits earned; that is, $\Pi_i = \psi_i \sum_t p_t y_t$. Note that this depends on both prices and the production plan of the firm.

An allocation being dynastically maximizing corresponds naturally to the dynasty using maximizing behavior given the resources it has available to it overall. Since there is a single dynastic budget set, it is as if the dynasty is fully free to make any transfers of wealth inside the dynasty that it chooses. Thus, an allocation being dynastically maximizing implies that no further transfers (e.g., bequests) within the dynasty can improve dynastic welfare (in a Pareto sense).

Next we define the analog of a competitive equilibrium among dynasties.

Definition 5 *(p^*, f^*, x^*, y^*) is a dynastic $\mathcal{P}$-equilibrium if*

1. *For all dynasties i, given (p^*, y^*), (f_i^*, x_i^*) is dynastically $\mathcal{P}$-maximizing.*

2. *(f^*, x^*, y^*) is feasible.*

3. *Given p^*, y^* maximizes profits, i.e. $p^* y \leq p^* y^*, \quad \forall y \in Y$.*

A Dynastic $\mathcal{A}$-equilibrium is defined similarly.

Theorem 1 *Suppose $u_i(x(i), f(i), f(-i), x(-i))$ is strictly monotone in $x(i)$ for all $i \in \mathcal{P}_0$ and that Assumptions 7 and 8 hold. If (p^*, f^*, x^*, y^*) is a dynastic $\mathcal{P}$-equilibrium, then $\sum_t p_t(\sum_{j \in \mathcal{P}_t \cap I(f)} e(j) + y_t^*) < \infty$, and (f^*, x^*, y^*) is $\mathcal{P}$-efficient. If (p^*, f^*, x^*, y^*) is a dynastic $\mathcal{A}$-equilibrium, then $\sum_t p_t(\sum_{j \in \mathcal{P}_t \cap I(f)} e(j) + y_t^*) < \infty$, and (f^*, x^*, y^*) is $\mathcal{A}$-efficient.*

[29] It is straightforward to extend these definitions to cover the case of external effects across dynasties.

Proof. See Appendix A.2.

The proof follows closely the logic of the regular proof of the first welfare theorem with two caveats. First, note that the usual first welfare theorem may fail in a regular OLG economy due to the double-infinity problem. This is not a problem here, because our equilibrium concept assumes that dynasties are maximizing, not individuals, and that the number of dynasties is finite. Secondly, for the case of $\mathcal{A}$-efficiency, the set of people that is 'eligible' to count in a potentially superior allocation is endogenous. Relatedly, the changing set of people could potentially cause problems when summing up over people. In Appendix A.2 we provide a detailed proof and show that these caveats do not cause problems here.

Summarizing the results from this section, we see that as long as each dynasty solves the internal redistribution problem efficiently, there are no external effects across dynasties, and all dynasties take prices as given, dynastic equilibria are efficient. In particular, fertility choices, and hence the sequence of populations that result, are efficient.

5 Dynastic Games and Efficiency

As is standard in models with external effects, equilibrium will naturally involve a mixture of price-taking behavior and quantity-taking behavior – the agent takes the prices it faces as fixed, and takes the actions, in particular the fertility choices of the other agents as fixed, when making its own consumption and fertility choices. Thus, the equilibrium notion is a mixture of Nash and Walrasian equilibrium.

Exactly what this means depends on the nature of the game being played by the agents, of course. The most straightforward treatment would be to formulate a game in which agents' choices are simultaneous moves chosen at time zero. One would then formulate the game in which the action of each agent included not only his own consumption and fertility choices, but also, possibly, a complex scheme of transfers to the other agents in his own dynasty. This game would generate a set of equilibrium strategy profiles, each of these generating an equilibrium outcome in terms of consumption and fertility decisions. Given the development in the sections above, the question would be, what types of games would generate equilibrium outcomes that are dynastically efficient (in either the $\mathcal{P}$ or the $\mathcal{A}$ sense)?[30]

Since fertility is intrinsically a dynamic decision, however, this is not the typical (or the best) way to model these types of decisions. Rather, models of fertility usually have a dynamic game theoretic formulation in which each agent who is born in period t must choose levels of both consumption and fertility in period $t+1$ as a function of all previous actions chosen by the preceding agents in his dynasty. These actions involve both the consumption and the fertility decisions of predecessors as well as the bequests left, etc.[31]

In this section we identify sufficient conditions for the equilibrium of the dynasty game to be efficient. We find that the degree of altruism and the richness of contracts between ancestors and descendants are crucial ingredients. Specifically, we argue that if dynasties are perfectly altruistic or if parents have perfect control over the actions of their descendants, then family games will lead to outcomes that are dynastically maximizing. The perfect altruism case includes the Barro-Becker model as a special case. The altruism eliminates the time inconsistency problem between parents and their

[30]From a formal point of view, this problem is similar to that studied in the clubs literature: When does a noncooperative formulation give rise to efficient outcomes? (See Scotchmer (1997) for an example.) However, the mechanism at work here is quite different. In club and other local public good environments, efficiency is guaranteed by competition between the clubs for members. Here, since the dynasty is the analog of a club, no such competition between clubs is possible. Rather, here the natural alignment of incentives within a family guarantees efficiency within the group.

[31]Ray (1987) and Streufert (1993) provide an explicit game-theoretic treatment of family interaction in the context of exogenous fertility and Raut (1992) in the context of endogenous fertility model with two-sided altruism.

descendants. Due to agreement between parents and children, contracts between parents and children can be fairly limited. We find that in this case, allowing for period-by-period bequests to a parents' own children is sufficient for efficiency. These bequests may need to be negative in some cases if the dynasties are sufficiently different.

A second extreme case that works requires no restrictions on preferences, but requires a rich set of bequest contracts. In particular, it is easy to see that if the head of the dynasty has a rich set of transfers that allows him to dictate the behavior of all descendants, then the time inconsistency problem becomes irrelevant.[32] This is a very extreme case, obviously. The point we want to emphasize here is that some combination of altruism and richness in bequests is needed to ensure that the equilibrium outcome of the game is efficient.

5.1 The Barro-Becker Model

One of the principle economic models of fertility is pioneered in Becker and Barro (1988) and Barro and Becker (1989). In this section, we show how our approach to efficient fertility can be applied to that class of models. In that approach, at each date, t, the individuals alive make decisions about their own consumption, how many children to have, and how large a bequest to leave each child. To make the model more tractable, Barro and Becker assume that fertility can take on any positive value, not just integers. Because of this, the analysis of the preceding sections does not directly apply to the Barro-Becker model. The modifications necessary are straightforward, however.[33]

We generalize the Barro-Becker framework here by allowing for more than one period 0 person. Each initial agent $i \in \mathcal{P}_0$ is the dynastic head of his own dynasty. We allow dynasties to differ in their initial capital stock, child-

[32] A formal analysis of this second benchmark case is available upon request.

[33] Details on this are available online in the technical appendix to this paper (Golosov, Jones and Tertilt (2006)).

rearing cost, discount factor, and per capita endowments (e.g. time). We also use a more general utility function. For most parts in this section, it is enough to focus on one dynasty. For these cases we drop the superscript i.

In the Barro-Becker model, it is assumed that each agent alive in period t, $i^t = (i^{t-1}, i_t)$, derives utility from his own consumption $x_t(i^t) \in \mathbb{R}^k$ and the utility of his children. Preferences of agent i^t are defined recursively by:

$$U_t(i^t) = u(x_t(i^t)) + \beta g(f_t(i^t)) \int_0^{f_t(i^t)} U_{t+1}(i^{t+1}) di_{t+1}.$$

Person i^t chooses his own consumption, $x_t(i^t)$, his fertility, $f_t(i^t) \in [0, \bar{f}]$ and a bequest vector for each of his children $b_t(i; i^t) \in \mathbb{R}^k$ subject to his own budget constraint:

$$p_t \left(x_t(i^t) + c(f_t(i^t)) \right) + \int_0^{f_t(i^t)} b_t(i_{t+1}; i^t) di_{t+1} \leq p_t e_t(i^t) + b_{t-1}(i_t; i^{t-1})$$

Note that $c_t(f_t(i^t))$ is childbirth costs in terms of the k goods and that the budget constraint includes the bequest that he has received from his own parents, $b_{t-1}(i_t; i^{t-1})$.

As before, we assume that the technology is characterized by a production set $Y \subset \mathbb{R}^{k\infty}$ and that the equilibrium production plan maximizes profits.[34]

Since our goal is to establish that an equilibrium is $\mathcal{P}$- and $\mathcal{A}$-efficient, when prices are determined by the interaction of multiple price-taking dynasties, we must first have a precise definition of what an equilibrium is. To do this, we will model the formulation above as an infinite horizon game in which in each period each child that is born must make decisions as given above. How then does a time t decision maker conjecture the future utility of his children? Of course, the answer is that they must correspond to the actual utility levels that these children receive if they optimally respond to the bequests that they receive from their parents, etc. That is, the sequence of consumption, fertility, bequest plans should be a subgame perfect equilibrium (SPE) of this infinite horizon game. Of course, there are typically many

[34]Throughout this section, we assume that Y is a convex cone containing 0 and hence ignore profits.

SPE's of infinite horizon games involving different threats of punishments off the equilibrium path. There is no easy way to select among these different equilibria, but one common selection criterion is that it not be too dependent on the assumption that time lasts forever. That is, it should be the limit of the equilibria of the finite horizon truncations of the infinite horizon game.

Definition 6 *An equilibrium is prices $\{p_t\}$, an allocation for each dynasty, $\{x_t^\tau(i^t), f_t^\tau(i^t), b_t^\tau(i; i^t)\}_{i^t,\tau}$ and a production plan $\{y_t\}$ such that:*

1. *For each dynasty τ, given $\{p_t\}$, $\{x_t^\tau(i^t), f_t^\tau(i^t), b_t^\tau(i; i^t)\}_{i^t}$ is the limit of the subgame perfect equilibrium outcome of the finite dynasty game (as described below).*[35]

2. *Given $\{p_t\}$, $\{y_t\}$ maximizes profits, i.e. $py \geq p\hat{y}, \quad \forall \hat{y} \in Y$*

3. *The allocation is feasible.*

We now describe the details of the finite dynasty game. A T horizon Barro-Becker game is a game in T+1 stages. The stages will be denoted by $t = 0, 1..., T$. In period 0, there is one player, player 0. His actions and preferences are denoted with 0 subscripts. In period t, $t \geq 1$, there are a continuum of players indexed by i^t, $i^t \in \mathcal{P}^t = [0, \bar{f}]^t$.

The strategy sets are as follows. In period 0, player 0 must choose

$$s^0 \in S^0 = \{(x_0, f_0, b_0(\cdot)) \mid p_0 (x_0 + c_0(f_0)) + \int_0^{\bar{f}} b_0(i) di \leq p_0 e_0\},$$

where $S^0 \subset R_+^k \times [0, \bar{f}] \times L_\infty^k([0, \bar{f}])$. Recursively, let h^{t-1} denote the history up to and including period $t-1$. In period t, $T > t \geq 1$, player i^t must choose

[35] Arguments similar to those in Fudenberg and Levine (1983) can be used to show that the limit of the SPE outcomes of the finite horizon truncations of this game are, themselves, SPE outcomes of the infinite horizon game. See Golosov, Jones and Tertilt (2006).

$$s^{i^t} \in S^{i^t}(h^{t-1}) = \begin{cases} A_t(h^{t-1}) & \text{if } i_t > f_{i^{t-1}} \\ \{(0,0,0)\} & \text{if } i_t \leq f_{i^{t-1}} \end{cases}$$

where $A_t(h^{t-1}) = \{(x_t(i^t), f_t(i^t), b_t(\cdot; i^t)) \mid p_t\,(x_t(i^t) + c_t(f_t(i^t))) + \int_0^{\bar{f}} b_t(i; i^t)di \leq p_t e_t(i^t) + b_{t-1}(i_t; i^{t-1})\}$. That is, if i^t is 'not born' he has no choices to make. In the case where i^t is born, $S^{i^t}(h^{t-1}) \subset R_+^k \times [0, \bar{f}] \times L_\infty^k([0, \bar{f}])$.

Finally, a player in period T makes similar choices except that he is constrained to choose $f_T(i^T) = 0$, and $b_T(\cdot; i^T) \equiv 0$.

Period 0 utility is given by:

$$U_0 = u_0(x_0) + \beta g_1(f_0) \int_0^{f_0} \left[u_1(x_1(i^1)) + \beta g_2(f_1(i^1)) \int_0^{f_1(i^1)} [u_2(x_2(i^2)) + ...] \right] di_T di_{T-1} ... di_1$$

Period t utility for player i^t is given by:

$$U_{i^t} = u_t(x_t(i^t)) + \beta g_{t+1}(f_t(i^t)) \int_0^{f_{i^t}} [u_{t+1}(x_{t+1}(i^{t+1}))$$
$$+ \beta g_{t+2}(f_{t+1}(i^{t+1})) \int_0^{f_{t+1}(i^{t+1})} [u_{t+2}(x_{t+2}(i^{t+2})) + ...]] di_T di_{T-1} ... di_{t+1}$$

A few technical assumptions are needed for our main result. For this, let $\mathcal{F}$ denote the set of all feasible sequences of total fertility and total consumption vectors.

Assumption 9 *1. Assume $c_t(f) = f c_t^*$ for some $c_t^* \in R_+^k$.*

2. $u_t(\cdot)$ is continuous, strictly increasing, strictly concave and $u_t(0) = 0$.

3. Assume $g_t(f) = f^\eta$ for some η.

4. Assume that $H_t(F, X) \equiv g_t(F) F u_t(X/F)$ is strictly increasing and strictly quasi-concave in (F, X).

5. Assume that utility is bounded on the feasible set – for some β, $\beta < \hat{\beta} < 1$, $\hat{\beta}^t H_t(F_t, X_t) \to 0$ for all $(F_t, X_t) \in \mathcal{F}$

We can now turn to the main result of this section.

Theorem 2 *Let the allocation* $z = \{\{x_t^\tau(i^t), f_t^\tau(i^t), b_t^\tau(\cdot; i^t)\}_{i^t,\tau}, y_t\}$ *and prices* $\{p_t\}$ *be a Barro-Becker equilibrium as defined in Definition 6. Then under Assumptions 9.1-5,* z *is* $\mathcal{P}$*-efficient and* $\mathcal{A}$*-efficient.*

The proof is given in Appendix A.3. The logic of the proof proceeds in four steps. First, we show that the equilibrium outcome of each dynastic game is unique. Second, we show that for each dynasty, the equilibrium outcome maximizes the utility of the period 0 player if he was choosing the allocation for the entire dynasty under a common budget constraint. Third, we show that it is the unique maximizer. This means that any other allocation that is affordable for the dynasty makes the dynastic head strictly worse off, which immediately implies that the allocation of the equilibrium outcome is dynastically $\mathcal{A}$- and $\mathcal{P}$-maximizing. The final step involves recognizing that all assumptions of Theorem 1 are satisfied and hence the equilibrium allocation is $\mathcal{A}$- and $\mathcal{P}$-efficient.

5.2 Discussion

Allowing for negative bequests may seem unusual. How crucial is this assumption for the result? Assume for a moment that there was a nonnegativity constraint on bequests, $b_t(i; i^t) \geq 0$. Note that if all dynasties were identical, then this constraint would never be binding in equilibrium; and hence, the equilibrium allocation would still be $\mathcal{P}$- and $\mathcal{A}$-efficient. If dynasties are heterogeneous, but not too different, then the same logic will apply by continuity. However, if the heterogeneity is big, then prohibiting negative bequests can indeed lead to an inefficiency in the Barro-Becker environment, as it effectively rules out certain mutually beneficial trades between parents and children.

Finally, note that if the model was extended to longer lifetimes, and parents and children would overlap for at least one period, then the nonnegativity of bequests could be replaced by the (more plausible) assumption that parents have some control over their children's resources.[36]

6 Applications

In many discussions, it is taken as a given by policy makers that fertility is 'too high' in developing countries and 'too low' in some developed countries.[37] Some governments provide free family planning and abortion services to discourage fertility, while others give large subsidies to encourage fertility. Few reasons are typically given for this view, although several auxiliary concerns are mentioned. These include the overall scarcity of factors as well as the role of population size and density in determining pollution.[38] In this section, we use the tools developed above to identify which of these concerns do and do not give rise to inefficient population growth rates. We find that scarce factors do not cause fertility to be inefficient, whereas global external effects do lead to inefficiencies. As pointed out in Section 3.1, there are typically never too many people in the $\mathcal{P}$-sense, and this will show up in some of the examples presented below.

6.1 Land Scarcity

In the policy debate it is often argued that because resources are scarce, fertility decisions affect society as a whole and should therefore not be left entirely to individuals. The logic provided is that parents do not take into account that an extra child decreases the amount of these scarce resources

[36]For notational convenience, we have assumed throughout the paper that people live for one period only. However, the logic of the proof of Theorem 2 does not depend on this assumption.

[37]See for example Financial Times (2004).

[38]Hardin (1968) argues that the "tragedy of the commons" leads to overpopulation. See also Becker and Murphy (1988) for a discussion of situations in which equilibria may be inefficient.

per capita. This leads to a discrepancy between private and social costs of children. Hence, an inefficiency might arise.[39]

In this section we argue that this logic is incorrect. The effect of reducing per capita income from adding an additional child (by increasing the aggregate labor supply) is analogous to the effect that an individual's increase in labor supply has on aggregate labor and thereby wages. These effects are channeled through prices and therefore do not lead to an inefficiency. Thus, this is an example of a *pecuniary externality*.

To see this, consider an example in which there are three goods in each period. The first is land, the second is time, and the third is a consumption good. All agents are endowed with one unit of time, which they supply inelastically to firms if they are born. Those agents alive in period 0, indexed by $i = 1, ..., N$, are also endowed with holdings of land, A^i. Let $\bar{A} = \sum_{i \in \mathcal{P}_0} A_i$ These holdings are sold to the firm and subsequently used forever. The production function is static: $y_t = F(A, \ell^f)$, where F is assumed to be constant returns to scale in land in labor input.

Profit maximization on the part of the firm then implies that the dynastic $\mathcal{P}$-equilibrium price of land traded in period 0 is given by $q_0 = \sum_t F_A(\bar{A}, N_t)p_t$, where N_t is the size of the population in period t and p_t is the equilibrium period 0 price of one unit of the consumption good in

[39]Many of those involved in the population debate are not economists. Because of this they do not carefully distinguish between true and pecuniary externalities. As a byproduct they often go back and forth between arguing that population is 'too high' simply because of crowding existing resources and because of taxing the ability of the environment to absorb pollutants. For an example, see the interview with Paul Ehrlich on Uncommon Knowledge where he states: " ... you're overpopulated when you no longer can live on your interest, when you've got to live on your capital. And the three main forms of capital that we're getting rid of very, very rapidly at today's density and today's consumption patterns are deep rich agricultural soils, biodiversity, which is critical, and maybe the most short-term critical is our supplies of groundwater everywhere, which are being overdrafted." See also Ehrlich (2002) and Dasgupta (2001) on crowding and population externalities.

period t. Similarly, the real wage rate must be $w_t/p_t = F_\ell(\bar{A}, N_t)$.

Thus, in keeping with intuition, if, for whatever reason, $\hat{N}_t > N_t$ for all t, and with p_t held fixed, the sale price of land (and the implicit rental price as well) is higher while the equilibrium real wage rate must be lower. That is, because land is scarce, if parents choose to have more children, real wages must be lower. In this sense, one parent would, across equilibria, lower the realized wage for all children by increasing his fertility choice. In this sense, there is crowding of scarce resources.

Despite this fact, it is easy to see that all of the assumptions of Theorem 1 are satisfied. It follows that the equilibrium fertility levels chosen will be $\mathcal{P}$-efficient (as well as $\mathcal{A}$-efficient) as long as individual dynastic decision making is done efficiently. Note that this result holds independent of the form of preferences. Thus, although the Barro-Becker formulation is one example in which this result is true, the conclusion is actually true more generally, as long as dynasties are maximizing.

6.2 Problems across Dynasties (Pollution)

Our theory also points to situations when equilibria are inefficient. The proofs of the first welfare theorems rely on the assumption that there are no external effects across dynasties. Many policy debates implicitly or explicitly question the validity of this assumption. In this section we discuss some of these arguments.

One of the most frequently discussed reasons for a negative effect of high population level is related to pollution and other adverse effects each agent may have on others. It is not clear, though, that such arguments justify policies that discourage fertility. For example, one might expect that standard Pigouvian taxes alone could restore optimality. In the Technical Appendix[40] we examine this issue in a context of a simplified two period version of the

[40] Golosov, Jones, Tertilt (2006)

Barro-Becker model where external effects arise from pollution as a byproduct of period 2 production.

We show that the equilibrium allocation without taxation is inefficient in two ways. First, there is 'too much' output in period 2 (in both an $\mathcal{P}$- and an $\mathcal{A}$ sense). This is the standard external effect. A standard Pigouvian tax on production leads to a Pareto improvement. It also achieves efficient allocations in the $\mathcal{P}$ sense. Even with this Pigouvian tax, however, the new allocations are not $\mathcal{A}$-efficient. The second inefficiency arises because the fertility is "too high". Each parent, by having children, adversely affects other parents through the pollution thereby created. This external effect is not internalized by the Pigouvian tax in the second period. Thus, endogenous fertility adds an additional dimension to the standard pollution problem – Parents exacerbate the pollution problem by having too many children, and a child tax, in addition to the Pigouvian pollution tax, will, in general, be needed.[41] Such a tax is not $\mathcal{P}$-dominating since it decreases the utility of children who are not being born because of the tax. This example shows that whether the fertility level is efficient with the pollution tax only, depends on the particular notion of efficiency one uses.

This reasoning needs to be adjusted if the direction of the external affects are reversed – for example, if they arise due to knowledge spillovers. A higher number of children is beneficial for both new and existing people, so that the equilibrium allocation without child subsidies is not only $\mathcal{A}$- but also $\mathcal{P}$-inefficient.

Yet another plausible externality could arise when there is heterogeneity in the degree of altruism towards one's children and some people derive disutility from seeing other parents neglect their children. It is easy to see that equilibrium fertility in such a case could be $\mathcal{A}$-inefficiently high, and that an $\mathcal{A}$-superior allocation would involve some people compensating others for

[41]This conclusion, and the example we analyze, is similar to that found in Harford (1998).

not having children. Alternatively, such an externality could provide an efficiency rationale for existing policies such as mandatory schooling, parental leave policies etc.

Other examples of the failure of the first welfare theorem in this environment arise when key markets are missing. One can imagine many examples relevant in fertility settings (for example, the lack of insurance against the risk of not being able to have children). A particularly interesting example involves private information about expected lifetimes. This is a common explanation given for the relative sparsity of annuity markets. This may lead parents to have too many children, because parents use children as an alternative to annuity contracts. In other words, an $\mathcal{A}$-superior allocation would involve fewer people with better insurance across dynasties. The missing markets problem is similar to the pollution externality discussed above. In both cases, dynasties may well be $\mathcal{A}$-maximizing, and yet, equilibrium fertility is too high due to a problem in the economy as a whole.

6.3 Problems within a Dynasty (Drugs)

We now give an example of a game among dynasty members that leads to an equilibrium outcome which is not dynastically $\mathcal{P}(\mathcal{A})$-maximizing. That is, this is an example showing that in certain contexts the assumption of dynastic maximization may not be an accurate description of real world fertility decisions.

There is one initial old person and one potential child, $\mathcal{P} = \{1, (1,1)\}$. The parent derives utility from her own consumption and from the consumption of her child: $u_1 = u(c_1) + f_1 \beta u(c_{(1,1)})$, where $u(\cdot)$ is strictly concave. The child has preferences over consumption, $c_{(1,1)}$, and drugs, $d_{(1,1)}$: $u_{(1,1)} = c_{(1,1)} + \gamma d_{(1,1)}$. People in each period are endowed with one unit of time. A static technology converts labor into consumption and drugs, $c + d \leq F(\ell) = w\ell$. It costs θ units of the consumption good to produce a child. Suppose $\gamma > 1$, then the optimal strategy for $(1, 1)$ is to consume only drugs, if born. Then the following is a sub-game perfect equilibrium allocation: $z = \{c_1 = w, f_1 = 0, c_{(1,1)} = 0, d_{(1,1)} = 0\}$. The reason for zero equilibrium fertility is that knowing that his child will be a drug addict, the parent prefers not to have a child. But note that, assuming θ is not too large, z is not $\mathcal{P}$-efficient, since the following allocation is $\mathcal{P}$-superior: $Z = \{c_1 = w - \theta, f_1 = 1, c_{(1,1)} = w, d_{(1,1)} = 0\}$.[42]

Note that the above inefficiency does not disappear with negative bequests. Instead, a tax-and-transfer system is required so that the parent can discourage the use, by the child, of the good the parent does not want the child to consume. More subtle disagreements between generations can cause similar problems. A very natural form of dissent would arise if parents and grandparents differ in their evaluation of their child/grandchild.[43]

Note, however, that time inconsistent preferences between parents and children do not have to lead to an inefficiency. It is easy to construct an example where parents and children disagree, but the equilibrium is still efficient, as any other allocation would make the child worse off. This point is related to an argument made in Section 3.1, where it was shown that efficiency need not coincide with utility maximization of the parent. Disagreement between parents and children may simply lead to an equilibrium allocation that favors the child (since the child chooses second), but this need not be inefficient.

7 Conclusion

In this paper, we have presented two extensions of the notion of Pareto-optimality for models in which fertility is endogenous, $\mathcal{P}$-efficiency and $\mathcal{A}$-

[42]The alternative allocation is also $\mathcal{A}$-superior.

[43]An example of this type, but with exogenous fertility, is given in Phelps and Pollak (1968)

efficiency. We have shown that, although models of fertility always have external effects, if these are confined to the family and the family makes optimal decisions, the time series of populations that is generated is optimal. One interesting implication of this result is that the Samuelson inefficiency that can be found in standard OLG economies disappears in this context. We have shown that the most popular economic model of fertility choice, that of Barro and Becker (1989), satisfies the assumption of dynastic optimization, and hence, in that model, population is efficient. Finally, we have shown that the presence of external effects can cause individually optimal fertility choices to be suboptimal from a social point of view and that this bias depends on the direction of the external effect.

Our analysis suggests the following typology for inefficiencies when fertility is endogenous.

1. The assumptions of the first welfare theorem might not be satisfied for standard reasons based on interactions among individuals. Examples include external effects, public goods, congestion effects, missing markets, and private information.

2. Limitations on bequests, lack of perfect altruism, and so on, cause the family allocation to not be $\mathcal{P}$-maximizing (or $\mathcal{A}$-maximizing).

There are several issues that have not been addressed in the current paper, but seem interesting for future research. One is to extend the concepts to allow for uncertainty and then analyze interactions between fertility and missing markets (such as annuity markets) in a more serious way. Secondly, this paper assumes unisexual reproduction, whereas one would like to be able to address questions of marriage. If marriage was endogenous, then dynasties could intermingle and potentially the whole world would be one dynasty. Finally, we think that an analysis of existing fertility policies would be very interesting. The results in this paper could be interpreted as saying that failures of intra-family coordination are more important than inter-family problems. If this was true, then one might want to correct any population problem by broadening the contract space between family members (i.e. richer inheritance law etc.) instead of giving out free contraception etc.

Only by pursuing this line of research can positive progress can be made into the important policy debates on population that are now being waged. As an example, some researchers argue that fertility is 'too low' in many European countries. The arguments typically given are along the line that the social benefit of having children exceeds the private one, because, without children, labor supply will be 'too small' in the future. This does not point to any particular reason for the theorems we have presented to not hold – no global external effects, or particular difficulties for families to be making efficient decisions are mentioned, etc., – and thus, it is reminiscent of the scarce factor example discussed above.[44] In environments without problems like these, the resulting allocation would be both $\mathcal{P}$ and $\mathcal{A}-$efficient and so no interventions are called for[45]. Even with problems like these, the appropriate intervention depends on the exact nature of the imperfection. Thus, while it is possible that the conclusion is correct – perhaps because of the difficulty in leaving negative bequests – we believe that it is critical to precisely identify the source of the inefficiency before a serious policy debate can be held.

[44]It also ignores that there are many places in the world where fertility is quite high, so that there seems to be little danger of labor supply being 'low' any time in the near future.

[45]Of course, another rationale for intervention is that it allows governments to choose a different efficient allocation from the one that arises in equilibrium.

A Appendix

A.1 Proof of Proposition 3

This proof uses the First Welfare Theorem (Theorem 1) in its construction. See Section A.2 for a proof of that result.

For any given population, I, let $C(I)$ be the total cost of child rearing with that population. Let $Y(I)$ be the total resources available for consumption when the set of people alive is I. That is:

$$Y(I) = \sum_{i \in I} e(i) - C(I)$$

Let $Y^j(I)$ be the total resources available for consumption if we consider only the endowments of a dynansty j in the population I, that is, $Y^j(I) = \sum_{i \in D_j \cap I} e(i) - C^j(D_j \cap I)$, where $C^j(D_j \cap I)$ is the total cost of rearing the children born to dynasty j if I is the population. Since child rearing costs are assumed additive across dynasties, $Y(I) = \sum_j Y^j(I)$.

Consider any $\mathcal{A}$-efficient allocation (f^*, x^*). Let (f_j^*, x_j^*) be the allocations in (f^*, x^*) that agents in dynasty j receive. By Assumption 6, there are no external effects across dynasties, and hence, for some wealth redistribution $T^* = (T_j^*)_{j \in \mathcal{P}_0}$ with $\sum_{j \in \mathcal{P}_0} T_j^* = 0$, the (f_j^*, x_j^*), $j \in \mathcal{P}_0$, each solve the dynastic maximization problem:

$$V^j(I(z^*), T^*, u^*) = \max_{(f,x)} u_j(f, x)$$

s.t.

$$u_i(f, x) \geq u_i^* \text{ for all } i \in I(f^*) \cap D_j \backslash \{j\}$$

$$\sum_{i \in I(z^*) \cap D_j} x_i \leq Y^j(I(z^*)) + T_j^*$$

for $u_i^* = u_i(f_j^*, x_j^*)$. We also let $u_i^* = \bar{u}_i$ for all $i \in \mathcal{P} \backslash I(f^*)$. We denote the vector of utilities arising in this way by $V(I(f^*), T^*, u^*) = (V^j(I(f^*), T^*, u^*))_{j \in \mathcal{P}_0}$.

Denote by $\alpha_j^* = (\alpha_i^*)_{i \in I(z^*) \cap D_j \backslash \{j\}}$ the vector of multipliers on the utility constraints on the problem (i.e., α_i^* is the multiplier on the constraint $u_i(f,x) \geq u_i^*$), and note that the this problem can be rewritten as maximizing a weighted sum of utilities of those dynasty members in $I(f^*) \cap D_j$ with weights given by α_i^* for the members in $I(f^*) \cap D_j \backslash \{j\}$ and 1 for j himself.

Lemma 1 *Consider any (f^*, x^*) which is in $\mathbb{A} \backslash \mathbb{P}$. Then there exists another population I, $I(f^*) \subset I$ and an allocation $(\tilde{f}, \tilde{x})$ that solves*

$$\max u_j(f,x)$$

s.t.

$$u_i(f,x) \geq \hat{u}_i \text{ for all } i \in I \cap D_j \backslash \{j\}$$

$$\sum_{i \in I(f^*) \cap D_j} x(i) \leq Y^j(I(f^*)) + T_j^*$$

Moreover, the solution to this problem is such that $V^j(I(f^*), T^*, u^*) = V^j(I, T^*, u^*)$ for all j.

Proof. Follows from Proposition 2 and discussion above. ∎

Pick any $(f^*, x^*) \in \mathbb{A} \backslash \mathbb{P}$ and corresponding α^*,T^*. Let $I^* = I(z^*)$ be the population in that allocation. Let $(\hat{f}, \hat{x})$ be any allocation that $\mathcal{P}$-dominates (f^*, x^*) such that condition a of Proposition 3 is satisfied. We know that $I(f^*) \subset I(\hat{f})$. From the lemma, we have that

$$V^j(I^*, T^*, u^*) = V^j(I(\hat{f}), T^*, u^*)$$

for all $j \in \mathcal{P}_0$.

Note that any allocation $(\hat{f}, \hat{x})$ with a population larger than I^* must have less total resources available for consumption, $Y(I(\hat{f})) < Y(I(f^*))$. Otherwise, all agents in $I(f^*)$ could receive exactly the same consumption as under (f^*, x^*), and this new allocaiton would clearly $\mathcal{A}$-dominate (f^*, z^*). This implies that there must be some agent with a positive α^* weight such that $x^*(i) > \hat{x}(i)$. Using our assumption about utility functions, Assumption 6, this implies that the consumption of all agents with positive α_i^* in the dynasty also falls. To see this, consider any agent with positive α_i^* weight. By Proposition 2, utilities of all other agents in the dynamisty either remain constant or increases. For his utility level to remain unchanged it must therefore be true that his consumption decreased.

Using the form of the utility function and the assumption that $x^*(i) > 0$ for all i, we can apply the envelope theorem:

$$V_{T_j}^j(I^*, T^*, u^*) = \sum_{i' \in D_j \cap I^*} \alpha_{i'}^* \frac{\partial u_{i'i}(x^*(i))}{\partial x(i)} < \sum_{i' \in D_j \cap I^*} \alpha_{i'}^* \frac{\partial u_{i'i}(\hat{x}(i))}{\partial x(i)} = V_{T_j}^j(I(\hat{f}), T^*, u^*), \tag{4}$$

since $u_{i'i}$ is strictly concave and $x^*(i) > \hat{x}(i)$.

Now we are ready to prove the main result:

Proof of Proposition 3: Assume that dynasty j^* has new people under the allocation $(\hat{f}, \hat{x})$, i.e. $I(\hat{f}) \backslash I(f^*) \cap D_{j^*} \neq \emptyset$. Assume that (f_j^*, x_j^*) is supported by the transfers T_j^* and that the allocation maximizes the social welfare function with weights α^*. Take a sequence of T_n converging to T^* with the restriction that $T_{j^*n} < T_{j^*}^*$ for all dynasties with more people under $(\hat{f}, \hat{x})$ and $\sum_j T_{jn} = 0$. Consider all the possible population sizes I with $I^* \subset I$. Since (f^*, x^*) is $\mathcal{A}$-efficient, it must be true that $V(I^*, T^*, u^*) \geq V(I, T^*, u^*)$ (here V^j is assumed to take the value $-\infty$ if the constraint set is empty in the maximization problem). Since $(f^*, x^*) \in \mathbb{A} \backslash \mathbb{P}$ this inequality must hold with equality for some I. Note that V is continuous in T at (I^*, T^*, u^*) as long as $\sum_{i \in D_j} x^*(i) > 0$ for all $j \in \mathcal{P}_0$, which is true by assumption. This implies that if $V^j(I^*, T^*, u^*) > V^j(I, T^*, u^*)$ for some j, I then $V^f(I^*, T_n, u^*) > V^j(I, T_n, u^*)$ for all T_n close enough to T^*. Therefore the allocations that solve $V^j(I, T_n, u^*)$ are $\mathcal{A}$-dominated by those solving $V^j(I^*, T_n, u^*)$. Consider any I such that $V^{j^*}(I^*, T, u^*) = V^{j^*}(I, T, u^*)$. By construction $T_{j^*n} < T_{j^*}^*$. Thus, in a neighborhood of T^*, using (4), we have, $V^{j^*}(I^*, T_n, u^*) > V^{j^*}(I, T_n, u^*)$ for all such j^*. Similarly, for all j such that $V^j(I^*, T^*, u^*) > V^j(I, T^*, u^*)$ it still true that $V^j(I^*, T_n, u^*) > V^j(I, T_n, u^*)$. It follows that the $(f_j(T_n), x_j(T_n))$, where $(f(T_n), x(T_n)) = (f_j(T_n), x_j(T_n))_{j \in \mathcal{P}_0}$, are dynastically $\mathcal{P}$-maximizing for each dynasty given the resources $Y^j(I(z^*)) + T_{jn}$. Thus, by the First Welfare Theorem, the $(f(T_n), x(T_n))$, are a sequence of $\mathcal{P}$-efficient allocations that have a population size I^*. Since $(f(T_n), x(T_n)) \to (f(T^*), x(T^*))$ this completes the proof.

A.2 Proof of Theorem 1

We provide the proof for $\mathcal{A}$-efficiency, the $\mathcal{P}$-efficiency proof is similar. It is useful to first prove the following lemma.

Lemma 2 *Assume $i \in P_0$ has strictly monotone preferences in $x(i)$. Let $(f^*(i), x^*(i))$ be dynastically $\mathcal{A}$-maximizing for dynasty D_i, given prices p and production y. Then $u_j(f(i), x(i)) \geq u_j(f^*(i), x^*(i))$ for all $j \in D_i$ implies that $\sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(f_i)} (x(j) + c(f(j))) \geq \Pi_i + \sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(f_i)} e(j)$.*

The proof of the lemma is very standard and hence omitted. One thing that is different from the usual proof is that with $\mathcal{A}$-maximization, the set of people that is eligible to count in an improving allocation depends on the original allocation. However, our assumption that the set of people at time 0 is fixed guarantees that this does not cause any problems.

We now proceed to prove Theorem 1. First, note that since $u_i(f(i), x(i); f(-i), x(-i))$ is strictly monotone in $x(i)$ for all $i \in \mathcal{P}_0$, for the given allocation to be a dynastic $\mathcal{A}$-equilibrium, (f_i, x_i) must be dynastically $\mathcal{A}$-maximizing, and hence,

$$\Pi_i + \sum_t \sum_{j \in \mathcal{P}_t \cap I(f_i)} p_t e(j) < \infty, \text{ for all } i.$$

Summing over i gives $\sum_t p_t \sum_{j \in \mathcal{P}_t \cap I(f)} e_j + y_t^* < \infty$.

Now, (p^*, f^*, x^*, y^*) is a dynastic $\mathcal{A}$-equilibrium and by way of contradiction, assume that it is not $\mathcal{A}$-efficient. Then an alternative feasible allocation $(\hat{f}, \hat{x}, \hat{y})$, exists that is $\mathcal{A}$-superior to (f^*, x^*, y^*). That is, $u_j(\hat{f}, \hat{x}) \geq u_j(f^*, x^*)$ for all $j \in I(f^*) \bigcap I(\hat{f})$ and $u_{j^*}(\hat{f}, \hat{x}) > u_{j^*}(f^*, x^*)$ for at least one $j^* \in I(f^*) \bigcap I(\hat{f})$. Assume $j^* \in D_{i^*}$. Then, since $(f^*_{i^*}, x^*_{i^*})$ is dynastically $\mathcal{A}$-maximizing, and since there are no external effects across dynasties (Assumption 8), it must be that $(\hat{f}_{i^*}, \hat{x}_{i^*})$ was not affordable for dynasty i^*, i.e.

$$\sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f}^*_i)} (\hat{x}(j) + c(\hat{f}(j))) > \Pi_{i^*} + \sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f}^*_i)} e(j)$$

Moreover, by Lemma 2, we know that for all other dynasties, i, the following must hold:

$$\sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f}_i)} (\hat{x}(j) + c(\hat{f}(j))) \geq \Pi_i + \sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f}_i)} e(j)$$

Summing over all dynasties, we get

$$\sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} (\hat{x}(j) + c(\hat{f}(j))) > \sum_t p^*_t [y^*_t + \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} e(j)] \ . \tag{5}$$

Note that the right hand side is finite; hence, the strict inequality is preserved. Profit maximization implies that $p^* y^* \geq p^* y$ for all other production plans $y \in Y$. Using this, we can rewrite equation (5) as

$$\sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} (\hat{x}(j) + c(\hat{f}(j))) > \sum_t p^*_t [\hat{y}_t + \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} e(j)] \ . \tag{6}$$

Finally, feasibility of $(\hat{f}, \hat{x}, \hat{y})$ implies that

$$\sum_{j \in \mathcal{P}_t \cap I(\hat{f})} (\hat{x}(j) + c(\hat{f}(j))) \leq \hat{y}_t + \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} e(j) \text{ for all } t$$

Multiplying the above by p^*_t and summing over t gives

$$\sum_t p^*_t \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} (\hat{x}(j) + c(\hat{f}(j))) \leq \sum_t p^*_t [\hat{y}_t + \sum_{j \in \mathcal{P}_t \cap I(\hat{f})} e(j)]$$

But this contradicts equation (6) which completes the proof. □

A.3 Proof of Theorem 2

As a first step in the proof of the theorem, we characterize the subgame perfect equilibrium outcomes of the finite horizon truncations of the game played inside a dynasty for a fixed set of prices. For this, let $\Gamma(a, q, T)$ denote the game with $T + 1$ periods, and initial dynasty wealth a when the prices are $q = (q_0, ..., q_T)$. Then, we have:

Lemma 3 *1. For every (a, q, T), $\Gamma(a, q, T)$ has a unique subgame perfect equilibrium in pure strategies, and hence a unique subgame perfect equilibrium outcome.*

2. For every (a, q, T), the SPE outcome is symmetric, $(x_s(i), f_s(i), b_s(i, j)) = (x_s(i'), f_s(i'), b_s(i', j'))$, for all $0 \le s \le T$ for all $0 \le i, i' \le f_{s-1}$, and for all $0 \le j, j' \le f_s$.

3. For every (a, q, T), for every $0 \le s \le T$, and every history up to s, the outcome of the continuation subgame is unique, symmetric, depends only on the bequest given to each agent, $b_{s-1}(i_s; i^{s-1})$, and solves:

$$\max_{x_s, f_s, x_{s+1}, f_{s+1}, ..., x_T} U_s = u(x_s) + \beta g(f_s) f_s u(x_{s+1}) + \beta^2 g(f_s) f_s g(f_{s+1}) f_{s+1} u(x_{s+2}) + ...$$
$$s.t. \quad q_s [x_s + c(f_s)] + q_{s+1} f_s [x_{s+1} + c(f_{s+1})] + ... + q_T (f_{T-1} f_{T-2} ... f_s) x_T$$
$$\le q_s e_s + q_{s+1} f_s e_{s+1} + ... + q_T (f_{T-1} f_{T-2} ... f_s) e_T + b_{s-1}(i_s; i^{s-1})$$

4. For every (a, q, T), for every $0 \le s \le T$, and every history up to s, the utility, at the SPE equilibrium outcome of the continuation game, realized by the time s player, $U_s(b_{s-1}(i_s; i^{s-1}))$, is strictly concave in $b_{s-1}(i_s; i^{s-1})$.

Note: in 3 and 4, we have adopted the notation that $b_{s-1} = a$.

Proof: The proof of the lemma proceeds by induction on T, beginning with $T = 0$, that is, a 1 period game. For the $T = 0$ case, the proofs of 1-4 are straightforward, with 4, that $U_0(a)$ is concave in a, being a standard result from consumer theory since u is strictly concave.

Given that 1-4 hold for T, we must show that they hold for $T+1$. That 4 holds follows immediately from the induction hypothesis for any $0 \le s \le T$. It follows that given any choice of strategies by the time 0 player, $(x_0, f_0, b_0(i))$, the equilibrium outcome of the resulting continuation game is unique and the utility received by player i in period 1 is given by $U_1(b_0(i))$.[46] Thus, the time zero player must solve:

$$\begin{aligned} \max_{x_0, f_0, b_0(i)} U_0 &= u(x_0) + \beta g(f_0) \int_0^{f_0} U_1(b_0(i)) di \\ s.t. \;\; q_0 \left[x_0 + c(f_0)\right] + \int_0^{f_0} b_0(i) di &\le q_0 e_0 + a \end{aligned}$$

First, we show that the solution to this problem has $b_0(i) = b_0 \; \forall i$ for some b_0. To see this, suppose that x_0^*, f_0^*, and $b_0^*(i)$, is the optimal choice for the period 0 player, and assume to the contrary that $b_0^*(i)$ is not constant. Note that an alternative strategy that is feasible is (x_0^*, f_0^*, b_0) where $b_0 = \frac{1}{f_0^*} \int_0^{f_0^*} b_0^*(i) di$. This alternative strategy fixes $(x_0, f_0) = (x_0^*, f_0^*)$ but makes bequests equal to all children. Under this alternative strategy, the payoff he receives is:

$$U_0 = u(x_0^*) + \beta g(f_0^*) f_0^* U_1(b_0(i)) > u(x_0^*) + \beta g(f_0^*) \int_0^{f_0^*} U_1(b_0^*(i)) di = U_0^*,$$

the payoff of the supposed optimal strategy. The inequality is strict because $U_1(b_0(i))$ is a strictly concave function of $b_0(i)$ and b_0^* is assumed non-constant. This establishes that the SPE has the property that $b_0(i)$ is a constant. This together with 2 from the induction hypothesis for T period games shows that 2 holds for all $0 \le s \le T+1$ in $T+1$ period games.

Given this, it follows that the period-0 agent must solve:

$$\begin{aligned} \max_{x_0, f_0, b_0} U_0 &= u(x_0) + \beta g(f_0) U_1(b_0) \\ s.t. \;\; q_0 \left[x_0 + c(f_0)\right] + f_0 b_0 &\le q_0 e_0 + a \end{aligned}$$

By the time consistency of preferences, $U_0(x_0, f_0, x_1, f_1, ..., x_{T+1})$ $= u(x_0) + \beta g(f_0) f_0 U_1(x_1, f_1, x_2, f_{2,} ..., x_{T+1})$, and given that $U_1(b_0)$ solves

[46] We simplify the notation for the time 0 allocation, because there is only one time 0 player, and hence the allocation does not need to be indexed by i^0.

(from 3 of the induction hypothesis):

$$\max_{x_1, f_1, x_2, f_2, \dots, x_{T+1}} \quad U_1 = u(x_1) + \beta g(f_1) f_1 u(x_2) + \beta^2 g(f_1) f_1 g(f_2) f_2 u(x_3) + \dots$$

$$\begin{aligned} s.t. \qquad & q_1 \left[x_1 + c(f_1) \right] + q_2 f_1 \left[x_2 + c(f_2) \right] + \dots + q_{T+1} \left(f_T f_{T-1} \dots f_1 \right) x_{T+1} \\ \leq \quad & q_1 e_1 + q_2 f_2 e_2 \dots + q_{T+1} \left(f_T f_{T-1} \dots f_1 \right) e_{T+1} + b_0, \end{aligned}$$

a standard two-step budget approach shows that the solution to the problem of the time zero agent is the same as that from solving:

$$\max_{x_0, f_0, x_1, f_1, \dots, x_{T+1}} \quad U_0 = u(x_0) + \beta g(f_0) f_0 u(x_1) + \beta^2 g(f_0) f_0 g(f_1) f_1 u(x_2) + \dots$$

$$\begin{aligned} s.t. \qquad & q_0 \left[x_0 + c(f_0) \right] + q_1 f_0 \left[x_1 + c(f_1) \right] + \dots + q_{T+1} \left(f_T f_{T-1} \dots f_0 \right) x_{T+1} \\ \leq \quad & q_0 e_0 + q_1 f_0 e_1 \dots + q_{T+1} \left(f_T f_{T-1} \dots f_0 \right) e_{T+1} + a \end{aligned}$$

Following Alvarez (1999), it is more convenient to write this problem in aggregate form by making the substitutions that $F_0 = 1$, $F_t = f_{t-1} F_{t-1}$, and $X_t = F_t x_t$. In this notation, the equilibrium outcome of the game solves the following concave optimization problem, which we call (PAggT).

$$\begin{aligned} \max_{X_0, F_1, X_1 \dots} \quad & U_0 = u(X_0) + \beta g(F_1) F_1 u(X_1/F_1) + \beta^2 g(F_1) F_1 g(F_2) F_2 u(X_2/F_2) \\ & + \dots + \beta^{T+1} g(F_1) F_1 \dots g(F_{T+1}) F_{T+1} u(X_{T+1}/F_{T+1}) \\ s.t. \quad & q_0 \left[X_0 + c(F_1) - e_1 \right] + q_1 \left[X_1 + c(F_2) - F_1 e_1 \right] + \dots + q_{T+1} \left[X_{T+1} - F_{T+1} e_{T+1} \right] \leq a \end{aligned}$$

By Assumptions 9.2 and 9.4, this problem has a unique solution and the utility realized, $U_0(a)$ is strictly concave in a. Thus, this establishes 3 and 4 for $s = 0$ in a $T+1$ period game. Coupled with 3 and 4 from the induction hypothesis, it follows that 3 and 4 hold for all $0 \leq s \leq T+1$ in any $T+1$ period game.

The fact that the solution to this problem is unique for $s = 0$ in a $T+1$ period game implies that the SPE of the $T+1$ period game is unique and that the outcome is unique, establishing the validity of part 1 for $T+1$. This completes the proof of the Lemma. □

The next step in the proof of the Theorem is to show that the solution to the time 0 planner's problem is also symmetric and, because of this, solves, in aggregates, the same concave maximization problem as the SPE outcome.

Lemma 4 *For each* (a, q, T)*, the solution to the dynasty planners problem at time 0 is symmetric* ($x_t(i) = x_t(i')$, $f_t(i) = f_t(i')$ *for all* i, i', t*, and, in aggregates, solves (PAggT).*

After successive substitution, the unconstrained dynasty head maximization problem is:

$$\max_{x_0, f_0, x_1(i_1), f_1(i_1), ...} U_0 = u(x_0) + \beta g(f_0) \int_0^{f_0} U_1(i_1) di_1$$

$$= u(x_0) + \beta g(f_0) \int_0^{f_0} u(x_1(i_1)) di_1 +$$

$$\beta^2 g(f_0) \int_0^{f_0} g(f_1(i_1)) \int_0^{f_1(i_1)} u(x_2(i_1, i_2)) di_2 di_1 + ... +$$

$$\beta^T g(f_0) \int_0^{f_0} g(f_1(i_1)) \int_0^{f_1(i_1)} g(f_2(i_1, i_2)) ... \int_0^{f_{T-1}(i^{T-1})} u(x_T(i^{T-1}, i_T)) di_T di_{T-1} ... di_1$$

$$s.t. \qquad q_0 [x_0 + c(f_0)] + q_1 \left[\int_0^{f_0} [x_1(i_1) + c(f_1(i_1))] \, di_1 \right] + ...$$

$$+ q_T \int_0^{f_0} \int ... \int_0^{f(i^{T-1})} x_T(i^{T-1}, i_T) di_T ... di_1$$

$$\leq a + q_0 e_0 + q_1 e_1 \int_0^{f_0} 1 di_1 + ... + q_T e_T \int_0^{f_0} \int ... \int_0^{f(i^{T-1})} 1 \; di_{T+1} ... di_1$$

The proof that the functions x_t and f_t are optimally chosen to be constants are tedious but straightforward, mimicking the arguments given in Lemma 3 above. For example, the analysis of the third term in the objective function is representative. This term is: $\beta^2 g(f_0) \int_0^{f_0} g(f_1(i_1)) \int_0^{f_1(i_1)} u(x_2(i_1, i_2)) di_2 di_1$. Denote quantities at the optimum with stars, i.e., $x_2^*(i_1, i_2)$, etc. To see that $x_2^*(i_1, i_2)$ is (a.e.) chosen to be a constant independent of i_2, suppose that this is not the case and consider the alternative plan in which all other variables are left unchanged but:

$$\hat{x}_2(i_1, i_2) \equiv \bar{x}_2(i_1) = \frac{1}{f_1^*(i_1)} \int_0^{f_1^*(i_1)} x_2^*(i_1, i_2) di_2.$$

Since u is strictly concave, it follows that:

$$\int_0^{f_1^*(i_1)} u(\hat{x}_2(i_1, i_2)) di_2 = \int_0^{f_1^*(i_1)} u(\bar{x}_2(i_1)) di_2 =$$

$$f_1^*(i_1) u(\bar{x}_2(i_1)) \geq \int_0^{f_1^*(i_1)} u(x_2^*(i_1, i_2)) di_2$$

and this inequality is strict unless $x_2^*(i_1, i_2) = \bar{x}_2(i_1)(a.e.)$. Since $x_2^*(i_1, i_2) = \bar{x}_2(i_1)$ also satisfies the budget constraint (leaving everything else unchanged), it follows that we can assume that $x_2^*(i_1, i_2) = x_2^*(i_1)$ without loss of generality.

Given this, the second term in the objective function becomes:

$$\beta^2 g(f_0^*) \int_0^{f_0^*} g(f_1^*(i_1)) f_1^*(i_1) u(x_2^*(i_1)) di_1.$$

To see that f_1^* and x_2^* are constants, if this is not true, consider the alternative plan, $\hat{f}_1$ and $\hat{x}_2$ given by:

$$\begin{aligned} \hat{f}_1(i_1) &= \bar{f}_1 = \frac{1}{f_0^*} \int_0^{f_0^*} f_1^*(i_1) di_1, \\ \hat{x}_2(i_1) &= \bar{x}_2 = \frac{1}{f_0^*} \int_0^{f_0^*} x_2^*(i_1) di_1. \end{aligned}$$

Since the function $g(f) f u(x/f)$ is weakly concave and strictly quasiconcave, it follows that:

$$\begin{aligned} & \beta^2 g(f_0^*) \int_0^{f_0^*} g(\hat{f}_1(i_1)) \hat{f}_1(i_1) u(\hat{x}_2(i_1)) di_1 \\ = \ & \beta^2 g(f_0^*) f_0^* g(\bar{f}_1) \bar{f}_1 u(\bar{x}_2) \geq \beta^2 g(f_0^*) \int_0^{f_0^*} g(f_1^*(i_1)) f_1^*(i_1) u(x_2^*(i_1)) di_1. \end{aligned}$$

Again, this equality is strict unless $(f_1^*(i_1), x_2^*(i_1)) = (\bar{f}_1, \bar{x}_2)(a.e.)$. This proposed change satisfies the budget constraint by construction (this uses the form of c). However, unlike in the step above, f_1 also enters the objective function (and the budget constraint) in other terms as well. Thus, to complete the proof, it is necessary to show that none of the other terms in the objective function are lessened by this change. Since this argument mirrors those given here step by step, this is not included. This shows that the solution to the time 0 planning problem problem is also given by the solution to (PAggT) and completes the proof of Lemma 4. □

Proof of Theorem 2:

Consider a Barro and Becker equilibrium. To show that the equilibrium allocation is $\mathcal{P}$- (respectively $\mathcal{A}$-) efficient, it is, by Theorem 1, sufficient to show that the equilibrium allocation is dynastically $\mathcal{P}$- (respectively $\mathcal{A}$-) maximizing at the given prices. This follows immediately once it is noted that the equilibrium allocation is the unique solution to the problem of the dynasty head, faced by the infinite horizon dynastic budget constraint. Any other allocation will make the dynastic head strictly worse off and hence, cannot be superior.

¿From the definition of equilibrium, an allocation that is part of a Barro and Becker equilibrium is, by assumption, the limit of a sequence of SPE equilibrium outcomes for the finite horizon truncations of the game (given prices). From the Lemmas, applied with $q = (p_0, p_1, ..., p_T)$ for each T, it follows that, for every T, this allocation is both unique, and solves the finite horizon truncated version of the dynasty heads maximization problem. By the definition of the Barro-Becker equilibrium allocation, it is the limit of the solutions to these finite horizon problems. The proof that this limiting allocation solves the limiting maximization problem is straightforward given assumption 6.5 and is omitted. □

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Figure 1

2 Period Example with $\bar{f}=3$

$\mathcal{P}$ = {1,2,(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)}
I(f)={1, 2, (1,1), (2,1), (2,2)}

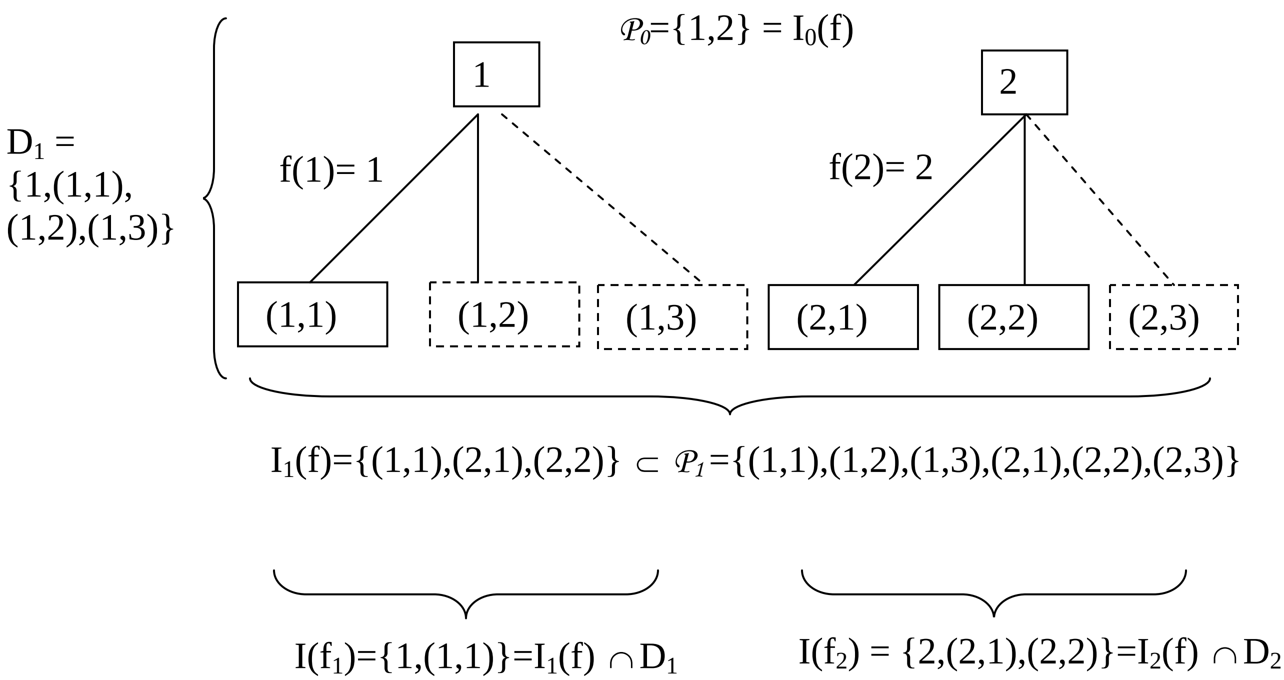

Figure 2

Assumption: u(e$_1$) > ū