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All-Pay Contests

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All-Pay Contests*

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Abstract

The paper studies a new class of games, “All-Pay Contests”, which capture general asymmetries and sunk investments inherent in scenarios such as lobbying, competition for market power, labor-market tournaments, and R&D races. Players have continuous, non-decreasing cost functions and compete for one of several identical prizes. The generality of players’ cost functions allows for differing production technologies, costs of capital, and prior investments, among others. I provide a closed-form formula for players’ equilibrium payoffs, and use it to compute aggregate expenditures, derive the effects of changes in contest structure, and analyze player participation. An algorithm for computing the unique equilibrium is given for a subclass of contests. This subclass nests multi-prize, complete-information all-pay auctions.

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1 Introduction

Many competitions are characterized by asymmetries among competitors. One example is the competition for promotions. Promotion decisions are often based on employees’ productivity and effort, as well as on tenure and quality of relationships with superiors. Thus, differences in employees’ abilities, social skills, and costs of effort translate to asymmetries in competition. Another example is the competition for rents in a regulated market. Since firms often compete by engaging in lobbying activities, firm-specific attributes such as quality and quantity of political connections, cost of capital, and geographic location frequently affect competition. A third example is research and development (R&D) races. The outcome of such races depends on firms’ research technologies, past R&D investments, and access to human capital, all of which may vary across firms. In addition to the asymmetries among competitors, each of the examples above has an “all-pay” feature: participating competitors make irreversible investments before the outcome of the competition is known.

This paper examines all-pay contests (henceforth: contests), which accommodate general asymmetries in an all-pay framework. Players compete for one of several identical prizes. Each player chooses a costly “score” and pays the associated cost. This all-pay component represents sunk investments and unconditional commitment of resources undertaken in the course of the competition. The players with the highest scores obtain one prize each. Asymmetries among players are captured by weakly increasing, continuous cost functions and by player-specific valuations for a prize. Cost functions and valuations are commonly known.

The special case of linear costs is known as an all-pay auction with complete information (henceforth: all-pay auction). It has been used to investigate competitions for promotions (Clark & Riis 1998), rent-seeking and lobbying activities (Hillman & Samet (1987), Hillman & Riley (1989), Baye, Kovenock & de Vries (1993)), competitions for a monopoly position (Elingsen 1991), waiting in line (Clark & Riis 1998), sales (Varian 1980), and R&D races (Dasgupta 1986). The empirical applicability of the all-pay auction is limited, however, in that asymmetries among competing parties are captured only by differences in valuations for a prize.

More general asymmetries have been analyzed in two-player, one-prize settings. Lazear & Rosen (1981) considered asymmetric costs of effort. Che & Gale (1998) investigated the effect of lobbying caps in two-player all-pay auctions. More recently, Kaplan & Wettstein (2006) and Che & Gale (2006) extended this analysis by using strictly increasing, ordered costs. Che & Gale (2003) considered the optimal design of one-prize research competitions under similar assumptions on players’ costs. In contrast, the class of games examined
here allows for multiple players, multiple prizes, and non-ordered costs. Such costs arise naturally in many competitive situations (see the example of Section 1.1 below).

The key result of the paper provides a closed-form formula for players’ expected equilibrium payoffs in a “generic” contest for \( m \) prizes. Expected payoffs are determined by reaches, powers, and the threshold. A player’s reach is the highest score he can choose without expending more than his valuation for a prize. The threshold is the reach of player \( m+1 \) when players are ordered in decreasing order of their reach. A player’s power equals his valuation less his cost of choosing the threshold. Theorem 1 shows that under “generic” conditions - checked using players’ powers - each player’s expected payoff equals the higher of his power and zero. Thus, a generic contest has the same payoffs in all equilibria. Theorem 1 also shows that the number of players who obtain positive expected payoffs equals the number of prizes.

The payoff result allows to bound aggregate expected equilibrium expenditures. When all players have identical valuations for a prize, precise expected expenditures can be calculated. These may sometimes be as low as zero, contrasting the complete rent dissipation predicted by all-pay auctions.

I consider how players’ payoffs and aggregate expenditures are affected by changes in the number of players, the number of prizes, and prizes’ valuations. In each case, the new expected payoffs can be computed using the payoff formula. When valuations are identical, expected aggregate expenditures can also be computed, informing optimal contest design. For example, adding a prize always makes players better off, whereas increasing prizes’ values may increase or decrease players’ expected payoffs.

I also examine the implications of the payoff result for equilibrium participation. Theorem 2 shows that for any number \( m \) of prizes, and any number \( k > m \) of players, there exist generic contests in which precisely \( k \) players participate (invest positively). In contrast, precisely \( m+1 \) players participate in an all-pay auction with distinct valuations. Participation is related to the ordering of players’ costs. For example, when players’ costs are strictly ordered, at most \( m+1 \) players participate in any equilibrium.

Participation is also linked to equilibrium uniqueness. When players’ costs are strictly increasing, the \( m+1 \) players with the highest powers must participate in every equilibrium. If additional players participate, multiple equilibria may exist. Conversely, Theorem 3 shows that under mild conditions, which include strictly increasing costs, a contest has at most one equilibrium in which precisely \( m+1 \) players participate. I provide an algorithm that computes the unique candidate for this equilibrium. I also give sufficient conditions for precisely \( m+1 \) players to participate in any equilibrium of a contest, and therefore for the equilibrium constructed by the algorithm to be the unique equilibrium. A special
case are all-pay auctions that satisfy the generic conditions. In their unique equilibrium, players’ strategies have interval supports. More generally, however, the support of a player’s strategy in the equilibrium constructed by the algorithm may consist of several disjoint intervals.

The rest of the paper is organized as follows. I begin with an example. Contests are defined in Section 2. The payoff characterization is derived in Section 3. Section 4 discusses participation and expenditures. Section 5 considers comparative statics. Section 6 describes the equilibrium construction algorithm. Section 7 concludes. Examples 1-4 are detailed in Appendix A. The proofs of the results of Sections 2, 3, and 4 are in Appendix B. The proofs of the results of Section 6 are in Appendix C.

1.1 An Example

Three lobbyists, each representing a firm, compete for a monopoly position of known value. Two lobbyists are energetic and relatively unknown, whereas the third is lazy and famous. The bureaucrat charged with allocating the monopoly position values the lobbyists’ reputation, and can also be influenced by “favors”.

This competitive situation can be modeled with the following contest, which is depicted in Figure 1. Players 1 and 2 are the energetic lobbyists; player 3 is the famous lobbyist. The famous lobbyist’s reputational advantage is captured by his initial marginal cost, \( \gamma \), which is low relative to the other lobbyists’ marginal costs. In contrast, his marginal cost for high scores is high relative to those of the energetic lobbyists.

The monopoly position is a prize of commonly known value 1. The cost functions are also commonly known. Each player chooses a score, and incurs the associated cost. The one choosing the highest score obtains the prize, with relevant ties broken randomly. The costs of choosing 1 for players 1,2, and 3 are \( K < 1 \), 1, and \( L > 1 \), respectively. Thus, players 2 and 3 would never choose a score higher than 1, since that would cost them more than the value of the prize. Player 1 can therefore guarantee himself a payoff arbitrarily close to \( 1 - K \) by choosing a score slightly higher than 1, so \( 1 - K \) is a lower bound on his expected payoff in any equilibrium.

These favors could be interpreted as rents transferred from the lobbyists to the bureaucrat, as expenditures that are socially beneficial, as wasteful activities, or as a combination of these. The interpretation does not affect the game, but is important for the normative implications.
Figure 1: Players' costs

Player 1 cannot, however, choose 1 with certainty in equilibrium, since players 2 and 3 would best-reply by choosing scores below 1, in which case player 1 would be better off choosing a lower score. In fact, player 1 must employ a mixed strategy in any equilibrium. It may therefore seem plausible that such a strategy could give player 1 an equilibrium payoff higher than $1 - K$.

Theorem 1 shows that the expected equilibrium payoff of player 1 is *exactly* $1 - K$. Similarly, players 2 and 3 can guarantee themselves no more than 0; the payoff characterization shows that this is exactly their equilibrium payoff. This implies that aggregate equilibrium expenditures equal $K$.

For low, positive values of $\gamma$, all three players must participate (invest) in equilibrium. This is shown in Section 4. Each player contributes to aggregate expenditures and wins the prize with positive probability. This participation behavior results from the non-ordered nature of players’ cost functions.\(^2\)

Now consider a variant of the contest, in which player 3 has marginal cost 0 up to score 1. This represents a very large initial advantage. In this case, there is a pure-strategy equilibrium in which player 3 wins with certainty and no player invests. Thus, it may be that no player invests in a contest for a valuable prize.

In both scenarios, precisely one player receives a positive expected payoff. This too follows from the payoff characterization, since there is only one prize.

Section 5 shows the effects of changes in competition structure. For example, the

\(^2\)Unlike in all-pay auctions, this does not rely on players’ valuations being identical (Baye et al. (1996)).
addition of player 3 to a contest that includes only players 1 and 2 changes neither expected payoffs nor expected aggregate expenditures, but changes individual expenditures. Thus, the addition of a player may change equilibrium behavior, without changing players’ payoffs or aggregate expenditures. In contrast, lowering the prize’s value can lead to a positive payoff for player 3, making him the only player who obtains a positive expected payoff.

2 The Model

There are \( n \) players, who compete for \( m \) homogeneous prizes, \( 0 < m < n \). The set of players \( \{1, \ldots, n\} \) is denoted by \( N \). Players compete by choosing a costly score, simultaneously and independently. The primitives of the contest are commonly known. This captures players’ knowledge of the asymmetries in abilities or technologies among them. Players’ equilibrium payoffs can therefore be thought of as "economic rents", in contrast to "information rents" that arise in models of private information.

Definition 1 An \( n \)-player, \( m \)-prize contest is a complete-information normal form game. Player i’s pure strategy space is \( S_i = [a_i, \infty) \) for initial score \( a_i \geq 0 \). Given scores \( s = (s_1, \ldots, s_n) \), \( s_i \in S_i \), player i’s payoff is \( u_i (s) = P_i (s) v_i - c_i (s_i) \) where:

1. Player i’s valuation for a prize is \( v_i \in (0, \infty) \).
2. Player i’s cost function \( c_i : S_i \to \mathbb{R}^+ \) is continuous and non-decreasing, with \( c_i (a_i) = 0 \) and \( \lim_{s_i \to \infty} c_i (s_i) > v_i \).
3. Player i’s probability of obtaining a prize is \( P_i (s) \). The \( m \) players with the highest scores win a prize, and relevant ties are resolved using any tie-breaking rule.

Note that the special case in which costs equal scores, initial scores equal 0, and ties are resolved by randomizing uniformly is a complete-information all-pay auction.

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\(^3\)When \( m \geq n \), all players obtain a prize with certainty and no player invests.

\(^4\)The resulting mixed-strategy equilibria can be interpreted as representing a player’s uncertainty about other players’ actions (Aumann (1987), Aumann & Brandenburger (1995)), or as a limiting case of incomplete information and finite-game approximations a-la Harsanyi (1973) and Reny (1999). (The purification results of Milgrom & Weber (1985) cannot be applied, since players’ utilities are discontinuous.)

\(^5\)\( c_i (a_i) = 0 \) is a normalization. Subtracting a constant from a player’s utility function does not change his strategic behavior, and shifts his equilibrium expenditures by the same constant.

\(^6\)I thank Eyal Winter for encouraging me to consider arbitrary tie-breaking rules.
Positive initial scores capture starting advantages, or “head starts”, without allowing players to choose lower scores. This may eliminate equilibria involving weakly dominated strategies that arise by setting costs to 0 on some initial range (see Example 4 in Appendix A).

As the example of Section 1.1 demonstrates, contests do not, in general, have pure-strategy equilibria. Existence of mixed-strategy equilibria is not immediately obvious, since payoffs are discontinuous in pure strategies, of which there is a continuum. Reny’s (1999) existence result can be applied to contests, showing the following (the proof of this result, and of the results of Sections 3 and 4 is in Appendix B).

**Proposition 1** Every contest has a Nash equilibrium in mixed strategies.

## 3 Payoff Characterization

The following concepts are key in analyzing the payoffs of players in equilibrium.

**Definition 2**

1. Player i’s **reach** is the highest score he can choose by expending an amount equal to his valuation. That is, \( r_i = \max_{x \in S_i} \{ c_i(x) = v_i \} \).\(^7\)

   Re-index players in (any) decreasing order of their reach, so that \( r_1 \geq r_2 \geq \ldots \geq r_n \).

2. Player \( m + 1 \) is the **marginal player**.

3. The **threshold** of the contest is the reach of the marginal player: \( T = r_{m+1} \).

4. Players i’s **power** equals his valuation less his cost of choosing the threshold. That is, \( w_i = v_i - c_i(\max \{ T, a_i \}) \). Thus, the marginal player’s power is 0.

In the example of Section 1.1, players are indexed in decreasing order of their reach, player 2 is the marginal player, and the threshold is 1. Player 1’s power is \( 1 - K > 0 \), player 2’s power is 0 and player 3’s power is \( 1 - L < 0 \).

In equilibrium, no player chooses scores higher than his reach with a positive probability, since choosing such scores leads to a negative payoff. Thus, a player with a positive power can guarantee himself a payoff arbitrarily close to his power by choosing a score slightly above the threshold. In addition, every player i can guarantee himself a payoff of 0 by choosing his initial score, \( a_i \). We therefore have the following.

\(^7\)\( r_i \) is well-defined since \( c_i \) is continuous, \( c_i(a_i) = 0 < v_i \), and \( \lim_{s_i \to \infty} c_i(s_i) > v_i \).
Lemma 1 A player’s expected payoff in any equilibrium is at least the maximum of his power and 0.

In fact, the payoff characterization states that expected equilibrium payoffs of contests that meet the following two conditions, discussed in Section 3.1, equal this lower bound.

Generic Conditions The marginal player is the only player with power 0 (Power Condition) and his costs are strictly increasing at the threshold (Cost Condition).\(^8\)

I refer to a contest that meets the Generic Conditions as a generic contest.

Theorem 1 In any equilibrium of a generic contest, the expected payoff of every player equals the maximum of his power and 0.

The following are immediate corollaries of Theorem 1.

Corollary 1 In a generic contest, players \(N_W = \{1, \ldots, m\}\) ("winning players") obtain strictly positive expected payoffs, and players \(N_L = \{m + 1, \ldots, n\}\) ("losing players") have expected payoffs of zero.

Proof. By Definition 2, players in \(N_W\) have non-negative powers, and players in \(N_L\) have non-positive powers. The Power Condition guarantees that players in \(N_W\) have strictly positive powers. 

Corollary 2 In a generic contest, a player obtains a strictly positive expected payoff if and only if his reach is strictly higher than the threshold.

Proof. Since players in \(N_W\) have strictly positive powers, their reaches are strictly higher than the threshold. By the Power Condition, players in \(N_L \setminus \{m + 1\}\) have strictly negative powers, and therefore have reaches strictly lower than the threshold.

Note that players’ equilibrium strategies may be mixed, so players in \(N_W\) may obtain a prize with probability smaller than 1, and players in \(N_L\) may obtain a prize with positive probability. Thus, it is only expected payoffs that are positive for players in \(N_W\), and 0 for players in \(N_L\).

Sketch of the proof of Theorem 1. Take a generic contest and an equilibrium of the contest. Each player’s equilibrium strategy is a probability distribution that assigns probability 1 to his set of best responses, i.e., scores that give the player his equilibrium.

\(^8\)i.e. for every \(x \in [a_{m+1}, T)\), \(c_{m+1}(x) < c_{m+1}(T)\).
expected payoff when other players play their equilibrium strategies. To pin down a player’s equilibrium payoff, it is therefore enough to identify a player’s payoff at one best response.

Given Lemma 1, it suffices to show that players in $N_L$ obtain at most 0, and players in $N_W$ obtain at most their power. The former is implied by the Zero Lemma below; the latter is shown by the Threshold Lemma below.

The Zero Lemma uses the following property of contest equilibria.

**Tie Lemma**  *If more than one player has an atom at a score $x$, i.e., chooses $x$ with positive probability, then all players who have an atom at $x$ either win with certainty or lose with certainty when choosing $x$.*

The intuition is that if more than one player has an atom at $x$ and not all players with an atom at $x$ win with certainty when choosing $x$, then some player with an atom at $x$ who does not win with certainty when choosing $x$ can increase his probability of winning a prize discretely by choosing a score slightly higher than $x$. This argument does not hold if all players with an atom at $x$ lose with certainty when choosing $x$, since then all players with an atom at $x$ may still lose with certainty by choosing a score slightly higher than $x$. Of course, if a player wins with certainty when choosing $x$, he does not benefit from choosing a higher score. Thus, many players may have an atom at the same score; the Tie Lemma shows that no two players can have an atom at the same score “when it counts”, i.e., when those players do not either win with certainty or lose with certainty when choosing that score. This result helps establish which players have an expected payoff of 0.

**Zero Lemma**  *At least $n - m$ players have best responses with which they win with arbitrarily small probability. Thus, at least $n - m$ players have an expected payoff of at most 0.*

The proof is by contradiction. If there exist $m + 1$ players who do not have such best responses, consider the union of their best-response sets. At least one player must have best responses close to the infimum $s_{\text{inf}}$ of this union. This player must lose with near certainty to the other $m$ players, otherwise at least two of the $m + 1$ players have an atom at $s_{\text{inf}}$, contradicting the Tie Lemma. Thus, there are at most $m$ players with strictly positive payoffs (note that the Zero Lemma holds regardless of the Generic Conditions).

Lemma 1 and the Power Condition show that the $m$ players in $N_W$ have strictly positive expected payoffs. Therefore, the Zero Lemma and Lemma 1 imply that under the Power Condition the $n - m$ players in $N_L$ obtain expected payoffs of 0. Using this fact, I now
show that players in \( NW \) obtain at most their power.

**Threshold Lemma**  
*Either \( m \) or \( m + 1 \) players have best responses that approach or exceed the threshold. If there are only \( m \) such players, then their costs must be constant on some interval leading up to the threshold. Thus, players in \( NW \) have an expected payoff of at most their power.*

To prove this, I show that \( m \) or \( m + 1 \) players “choose scores up to (or above) the threshold”, i.e., choose scores higher than \( x \) with a positive probability, for every \( x < T \). This implies that these players have best responses that approach or exceed the threshold. Since players in \( NL \setminus \{m + 1\} \) have negative powers, their reaches are below the threshold. Therefore, they do not choose scores up to the threshold in equilibrium. Thus, only the \( m + 1 \) players \( NW \cup \{m + 1\} \) may choose scores up to the threshold. At least \( m \) of them must do so, or else the marginal player could win with certainty by choosing a score close enough to the threshold; because of the Cost Condition, this would give him a positive payoff, a contradiction (recall that the marginal player is in \( NL \)). If \( m + 1 \) players choose scores up to the threshold, they must therefore be players \( NW \cup \{m + 1\} \). If precisely \( m \) players choose scores up to the threshold, any one of them could choose a score slightly lower than the threshold and still win with certainty. This is not be a profitable deviation only if all of them have constant costs on some interval leading up to the threshold. Since the marginal player’s costs are increasing at the threshold, these \( m \) players must be all the players in \( NW \). In either case, all players in \( NW \) choose scores up to the threshold, so obtain at most their power.

The Zero Lemma shows that players in \( NL \) obtain at most 0, and the Threshold Lemma shows that players in \( NW \) obtain at most their power. Since \( NL \cup NW = N \), all players obtain at most the maximum of their power and 0. Lemma 1 shows the converse, establishing Theorem 1.

In the example of Section 1.1, \( NW = \{1\} \) and \( NL = \{2, 3\} \). The contest is generic, so player 1’s payoff is \( 1 - K \), and those of players 2 and 3 are 0.

### 3.1 Discussion of the Payoff Characterization

The generality of players’ cost functions may lead to complicated equilibria. Equilibrium payoffs, however, depend only on cost functions’ values at the threshold. From an applied perspective, only the reach of each player and his cost at the threshold need to be computed - it is not necessary to estimate the entire cost function.
The result does not rely on equilibrium uniqueness. Example 3 in Appendix A describes a generic contest with multiple equilibria, which lead to different allocations and aggregate expenditures. Therefore, standard revenue equivalence techniques cannot be used to compare players’ payoffs across equilibria.

A player’s equilibrium payoff in a generic contest coincides with his max-min value (the supremum of the payoffs he can guarantee himself regardless of other players’ strategies), after deletion of strictly dominated strategies. Since scores that are higher than a player’s reach are strictly dominated by his initial score, a player with positive power can win a prize with certainty by choosing a score slightly above the threshold. Thus, the max-min value of a player in $N_W$ equals his power. A players with a non-positive power cannot obtain a positive payoff if all players in $N_W$ choose the threshold. Therefore, the max-min values of of a player in $N_L$ is 0. Note that payoffs are as if players in $N_W$ chose the threshold and players in $N_L$ did not invest, even though, in general, this is not an equilibrium.

In contrast, the payoff of a player in a contest that does not meet the Generic Conditions may be arbitrarily close to his valuation, regardless of his power. Example 1 shows this when the Cost Condition fails; Example 2 shows this for the Power Condition. Contests that do not meet the Generic Conditions can be perturbed slightly to meet them: perturbing players’ valuations generates a contest that meets the Power Condition; perturbing the marginal player’s costs around the threshold leads to a contest that meets the Cost Condition. For example, in an all-pay auction the Cost Condition is met trivially, because costs are strictly increasing. If the Power Condition is met, i.e., the valuation of player $m+1$ is different from those of all other players, the all-pay auction is generic. In this case, since the reach of each player equals his valuation, the payoff of every player $i$ is max $\{v_i - v_{m+1}, 0\}$.

4 Implications of the Payoff Characterization

4.1 Participation

In the real world, more than $m+1$ contenders often participate in a competition for $m$ prizes even when participation is costly. Promotions in the workplace are a case in point:

9 The Zero Lemma and the Generalized Threshold Lemma (proven in Appendix B) can be used to extend the payoff result to certain contests for which the Power Condition does not hold. Such contests include all two-player contests with strictly increasing costs, as shown in Proposition 6 below, and some of the one-prize all-pay auctions of Baye et al. (1996).
competition for a single promotion commonly prompts effort by more than two workers. Similarly, more than two firms frequently lobby for one monopoly position.

What determines the number of active participants in a competitive situation? More than \( m + 1 \) competitors may participate if competitors are uncertain about their strength relative to others (see Krishna (2002)), or if there is some exogenous randomization, so that every competitor has a positive probability of winning (see Tullock (1980)). In contrast, precisely the \( m + 1 \) players with the highest valuations participate in a generic, complete-information all-pay auction.\(^{10}\) Thus, it is natural to ask whether participation by \( m + 1 \) players is a hallmark of complete-information, deterministic models of competition. The answer is no.

There exist generic contests in which any number of players “participate”, i.e., invest positively in expectation. It is straightforward to see that fewer than \( m + 1 \) players may participate when initial scores are higher than the threshold.\(^{11}\) When players’ costs are strictly increasing and all initial scores equal 0, at least \( m + 1 \) participate; this follows from the Threshold Lemma, which implies that players 1, \ldots, \( m + 1 \) play at least up to the threshold. In fact, more than \( m + 1 \) players may participate.

**Theorem 2** For any \( n, m \) and \( k \) such that \( n \geq k > m > 0 \), there exist generic contests with \( n \) players, \( m \) prizes, and strictly increasing costs starting at 0, such that in any equilibrium, precisely \( k \) players participate.

Intuitively, Theorem 2 stems from the difference between local cost advantages and players’ powers. The former determine participation; the latter determine payoffs. When players with negative powers have local cost advantages over those with non-negative powers, many players may participate, even though only the \( m \) players in \( N_W \) obtain positive payoffs. In the example of Section 1.1, the Threshold Lemma implies that players 1 and 2 choose scores up to the threshold, and so participate in any equilibrium. If player 3 did not participate, players 1 and 2 would have to play strategies that make both of them indifferent among all scores up to the threshold.\(^{12}\) For low values of \( \gamma \), player 3 could then

\(^{10}\)Baye et al. (1996) showed that more than two bidders may place positive bids in certain non-generic, one-prize all pay auctions.

\(^{11}\)Only one player invests in the following two-player contest for one prize of value 1. Player 1 has linear costs on \([0, \frac{1}{2}]\) and 0 marginal costs on \([\frac{1}{2}, 1]\). Player 2 has a head-start of \( \frac{1}{2} \) \( (a_2 = \frac{1}{2}) \) and very high marginal costs on \([\frac{1}{2}, 1]\), so that the threshold lies in \([\frac{1}{2}, 1]\). An equilibrium is for player 1 to play the threshold with certainty, deterring player 2 from participating. Note that this contest meets the Generic Conditions, and only player 1 plays up to the threshold.

\(^{12}\)See Proposition 6 below.
obtain a positive payoff by choosing a low score. Thus, player 3 must also participate in any equilibrium.

In contrast, if players with strictly negative powers are locally disadvantaged everywhere with respect to the marginal player, they do not participate in any equilibrium.

**Proposition 2** In a generic contest, if the normalized costs of the marginal player are strictly lower than those of a player \( i > m + 1 \), i.e.,

\[
\frac{c_{m+1} \left( \max \{a_{m+1}, x\} \right)}{v_{m+1}} < \frac{c_i(x)}{v_i} \quad \text{for all } x \in S_i \text{ such that } c_i(x) > 0
\]

then player \( i \) does not participate in any equilibrium. In particular, if the marginal player’s normalized costs are strictly lower than those of all players in \( N_{\setminus \{m+1\}} \), then in any equilibrium only players in \( N_{W} \cup \{m+1\} \) may participate.

Proposition 2 explains players’ participation behavior in generic all-pay auctions, which include all-pay auctions with distinct valuations. Players’ cost functions are strictly increasing, so players \( 1, \ldots, m + 1 \) choose scores up to the threshold; Proposition 2 shows that no other players participate. Note that the relation between cost functions of players in \( N \setminus \{m+1\} \) is not constrained by Proposition 2.

The sufficient condition of Proposition 2 is by no means necessary. There exist generic contests that fail to meet this condition, and in which only the most powerful \( m+1 \) players participate in any equilibrium.\(^{13}\)

### 4.2 Aggregate Expenditures

Given a generic contest and an equilibrium \( G = (G_1, \ldots, G_n) \), where \( G_i(x) \) is the probability that player \( i \) chooses a score smaller or equal to \( x \), denote by \( AE \) the (expected) aggregate equilibrium expenditures, by \( V \) the (expected) equilibrium allocation value and by \( U \) players’ aggregate (expected) payoffs. \( AE = \sum_{i=1}^{n} E_G(c_i(s_i)) \), \( V = \sum_{i=1}^{n} v_i E_G(P_i(s)) \) and \( U = \sum_{i=1}^{n} u_i \), where \( E_G \) denotes the expectation with respect to players’ equilibrium strategies \( (u_i \text{ does not depend on the equilibrium } G) \). Since a player’s equilibrium payoff equals his expected gains minus his expected costs, i.e., \( u_i = v_i E_G(P_i(s)) - E_G(c_i(s_i)) \), we obtain the expenditures formula \( AE = V - U \). This formula shows that if players’ valuations all equal \( v \) (for example, when the prizes are monetary), aggregate expenditures equal \( mv - U \) in all equilibria, and are pinned down by the payoff result.

\(^{13}\)One such contest is specified by \( n = 3, m = 1, v_1 = 2.95, v_2 = 1, v_3 = 2, a_i = \frac{(i-1)}{2} \), and linear costs.
When players’ valuations differ, aggregate expenditures are not pinned down by players’ payoffs. Nevertheless, non-trivial upper and lower bounds on $V$ can be given, which lead to bounds on aggregate expenditures. Each player’s equilibrium payoff provides a lower bound on his probability of winning a prize, leading to a lower bound on $V$. The efficient allocation gives an upper bound on $V$. These bounds, as well as players’ payoffs, are continuous in valuations. Thus, the estimation of aggregate expenditures when valuations are equal is robust to slight perturbations in valuations.

In an all-pay auction, as players’ valuations approach a common value, aggregate expenditures approach $V$, and players’ expected payoffs approach 0. In the limit, rent dissipation is complete (Clark & Riis (1998)). Other contests can generate different predictions, since payoffs are determined by players’ costs at the threshold. These costs may be as low as zero, leading to zero rent dissipation. This occurs in the example of Section 1.1 when player 3’s head start is sufficiently large. Alternatively, costs may approach players’ valuations even when valuations differ across players, leading to complete rent dissipation. Since payoffs are non-negative, expenditures are bounded above by the value of equilibrium allocations.

5 Comparative Statics

The payoff characterization illustrates the effects of changes in contest structure on players’ payoffs, regardless of differences in players’ valuations. I consider adding a player, adding a prize, and changing prizes’ values; other comparative statics, such as removing a player and changing players’ cost functions, can be derived similarly. When valuations are equal, the expenditures formula shows the effects of such changes on aggregate expenditures. The analysis provides empirical implications, and informs optimal contest design. In what follows, I consider changes that do not violate the Generic Conditions.

5.1 Adding a Player

Consider the effects of adding a player with cost function $c$, valuation $v$, and reach $r$. Regardless of the value of $r$, the threshold weakly increases. Thus, existing players’ payoffs weakly decrease: the addition of a player always makes existing players weakly worse off.

If $r$ is lower than the existing threshold, the new player has a payoff of 0, and the threshold does not change. Consequently, existing players’ payoffs do not change, though the equilibrium may change. If valuations are equal, aggregate expenditures, which equal
mv − U, do not change. Individual expenditures, however, may change.\textsuperscript{14}

If $r$ is higher than the existing threshold and lower than $r_m$, the new threshold is $r$ and the new player’s payoff is 0. If valuations are equal, aggregate expenditures weakly increase, taking a value in $[AE, mv]$, where $AE$ is the aggregate expenditures of the original contest. The exact value depends on the decrease in existing players’ payoffs: if $r$ is sufficiently close to the existing threshold, payoffs decrease very little and expenditures increase slightly; if $r$ is close to $r_m$ and players’ costs at $r_m$ are close to the prizes’ value, payoffs approach 0 so expenditures approach $mv$.

If $r$ is higher than $r_m$, the new threshold is $r_m$ and the new player obtains a positive payoff. If valuations are equal, aggregate expenditures take a value in $[AE − v, mv]$. The exact value depends on the new player’s payoff relative to the decrease in existing players’ payoffs: as $r$ approaches $r_m$ the new player’s payoff approaches 0; as $c(r_m)$ approaches 0 the new player’s payoff approaches $v$.

### 5.2 Adding a Prize

Consider adding a prize, assuming that the number of players in the contest exceeds $m + 1$. The marginal player of the new contest is player $m + 2$, so the threshold decreases to $r_{m+2}$. Thus, players’ payoffs weakly increase: the addition of a prize always makes players better off.

If the value of a prize is $v$ for all players, aggregate expenditures take a value in $[0, AE + v)$. The exact value depends on whether the aggregate increase in players’ payoffs is larger or smaller than $v$. If $r_{m+2}$ is sufficiently close to the existing threshold, players’ payoffs increase very little, and expenditures increase. If $r_{m+2}$ is much lower than the existing threshold, players’ payoffs may increase significantly. As $r_{m+2}$ approaches 0, positive powers approach $v$ and expenditures approach 0.

### 5.3 Increasing Prizes’ Values

Increasing prizes’ values can lead to re-indexing of players. Thus, the sets $N_W$ and $N_L$ may change, as may the set of participating players. In particular, the payoffs of some players may increase while those of others decrease. If all valuations are increased from $v$ to $v + \Delta$, aggregate expenditures weakly increase, taking a value in $[AE, m(v + \Delta)]$:

\textsuperscript{14}In the example of Section 1.1, adding player 3 with low $\gamma > 0$ to a contest that includes only players 1 and 2 changes neither payoffs nor aggregate expenditures. The equilibrium changes, since all three players must participate, as discussed above.
since the threshold weakly increases, the payoff of each of the \( m \) players now in \( N_W \) increases by no more than \( \Delta \), whereas the total allocation value increases by \( m\Delta \). As the threshold increases, winners’ payoffs approach 0, so expenditures approach \( m(v + \Delta) \). These predictions differ from those of all-pay auctions. In an all-pay auction, increasing all player valuations by the same amount leads to a commensurate increase in expenditures and does not change players’ payoffs.

### 5.4 Optimal Contest Design

When designing a contest, functions \( f(u_1, \ldots, u_n) \) of players’ payoffs can be optimized by choosing the optimal set of contenders, the number of prizes, and their value, using the above analysis. In addition, player-specific scoring rules may be implemented to affect players’ cost functions. If valuations are equal, functions \( g(u_1, \ldots, u_n, AE) \) of players’ payoffs and aggregate expenditures can also be optimized.

One example is choosing the optimal number of identical monetary prizes to maximize expenditures, given a set of players and a fixed budget \( V \).\(^{15}\) The allocation value of the contest equals \( V \), regardless of the number of prizes, so aggregate expenditures are maximized when the sum of players’ payoffs is minimized. The number of players who obtain a positive payoff equals the number of prizes; increasing the number of prizes lowers the threshold and increases the number of players who obtain a positive payoff. The value of each prize is lower, however, so aggregate payoffs may increase or decrease as the number of prizes increases. This depends on players’ costs at the thresholds that correspond to the various number of prizes.

A related question, given a contest for \( m \) prizes of equal value \( v \), is whether adding another prize of value \( v \) leads to higher expenditures than adding \( \frac{v}{m} \) to each of the existing prizes. As shown above, adding a prize may decrease aggregate expenditures, whereas increasing prizes’ value weakly increases expenditures. However, the latter does not dominate the former. The effects of the two alternatives on expenditures are seen by considering players’ costs at the new thresholds.

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\(^{15}\)Moldovanu & Sela (2001) considered the optimal allocation of prizes in competitions to maximize contestants’ expected efforts. Their framework models players that are ex-ante identical, and allows for non-identical prizes. In contrast, contests accommodate asymmetries between players and are restricted to identical prizes.
6 Solving for Equilibrium

Equilibrium allocations, scores, individual expenditures, and aggregate expenditures when valuations are different cannot be deduced from the payoff characterization alone. For these, solving for equilibrium is necessary. I consider regular contests, defined as follows.

Definition 3 An n-player, m-prize contest is a regular contest if it is generic and meets the following two regularity conditions:

1. The costs of all players are strictly increasing, and all initial scores equal 0.
2. The costs of players 1, . . . , m + 1 are piecewise analytical on [0, T].\(^{16}\)

Condition (1) and Threshold Lemma imply that players 1, . . . , m + 1 participate in every equilibrium of a regular contest. Example 3 shows that if additional players participate, multiple equilibria may exist. By solving for the unique equilibrium of m + 1-player regular contests, I show that n-player regular contests have at most one equilibrium in which precisely m + 1 players participate.

A key step in solving for equilibrium is identifying players’ equilibrium best-response sets, or strategy supports (recall that strategies are probability distributions). The difficulty is that best-response sets may not be intervals, as shown by the unique equilibrium of the three-player, two-prize, regular contest depicted in Figure 2. Figure 3 depicts players’ equilibrium strategies, drawn as cumulative probability distributions (CDFs).\(^{17}\)

In the equilibrium, the best-response set of player 2 is \((0, x_1] \cup [x_2, 1]\), and that of player 3 is \([0, x_3] \cup [x_4, 1]\). Each player is defined as being “active” on his best-response set, with the possible inclusion of 0. The algorithm described below constructs the equilibrium by identifying the active players on each interval (denoted in curly brackets), and the “switching points” above which the set of active players changes \((x_k, 1 \leq k \leq 4, and 1)\).\(^{18}\)

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\(^{16}\) A function \(f\) is piecewise analytical on \([0, T]\) if \([0, T]\) can be divided into a finite number of closed intervals such that the restriction of \(f\) to each interval is analytical. Analytical functions include polynomials, the exponent function, trigonometric functions, and power functions. Sums, products, compositions, reciprocals, and derivatives of analytical functions are analytical (see, for example, Chapman (2002)).

\(^{17}\) Players’ cost functions are given in Appendix C.4.

\(^{18}\) The problem of solving for an equilibrium can be formulated as an optimal control problem with linear control, in which the state variables are players’ CDFs. The objective is to minimize the sum of players’ expected payoffs, subject to the constraints that CDFs are non-decreasing and that no player obtains more than his power. This latter constraint is a state constraint, which precludes the application of
Figure 2: Player’s costs, reaches, and powers

Figure 3: The unique equilibrium

Pontryagin’s Maximization Principle in its standard form (see Hestenes (1966) and Seierstad & Sydsaeter (1977)). Thus, the standard “pasting conditions” cannot be used, and the same difficulties remain in determining the switching points and sets of active players.
For such construction to be possible, the equilibrium must be “well-behaved”, i.e., for every \( x < T \), the set of active players immediately above \( x \) must remain constant:

**Definition 4** An equilibrium is **constructible** if for every score \( x < T \) there exists some \( \bar{x} > x \) such that for each player either every score in \( (x, \bar{x}) \) is a best response, or no score in \( (x, \bar{x}) \) is a best response.

The algorithm solves for a constructible equilibrium of an \( m+1 \)-player regular contest. After describing the algorithm, I show that every equilibrium of the contest, constructible or not, must coincide with the one constructed, which is therefore the unique equilibrium.

### 6.1 Equilibrium Construction

I derive necessary conditions for a constructible equilibrium \( G = (G_1, \ldots, G_{m+1}) \), where \( G_i(x) \) is the probability that player \( i \) chooses a score smaller or equal to \( x \). These conditions show that \( G \) is continuous above 0, and that \( G(0) \) is determined by players’ powers. Also, for every \( x \in [0, T) \) the conditions provide an explicit procedure for defining \( G \) on \( [x, x_k] \), which relies only on the value \( G(x) \), where \( x_k \) is the first switching point higher than \( x \), i.e., the first score above which the set of active players changes. The algorithm constructs \( G \) by using this procedure and proceeding from 0 to \( T \). Since players’ costs are strictly increasing, the Threshold Lemma and the payoff characterization imply that all players choose scores exactly up to the threshold. It therefore suffices to specify \( G \) on \( [0, T] \). I assume that valuations equal 1: Lemma 7 in Appendix B shows that dividing a player’s valuation and cost function by his valuation does not change the set of equilibria.

To begin, suppose that \( G(x) \) is known for some \( x \in [0, T) \), and let \( y \in (x, T) \). Consider player \( i \) for whom \( y \) is a best response. By choosing \( y \), player \( i \)’s expected benefit equals his probability of winning a prize since his valuation is 1. His cost of choosing \( y \) is \( c_i(y) \). Since there are \( m+1 \) players, payoffs equal powers so \( y \) is a best response for player \( i \) if and only if

\[
P_i(y) - c_i(y) = w_i
\]

where \( P_i(y) \) is the probability that \( i \) wins a prize when other players choose scores according to \( G \). Equivalently,

\[
1 - P_i(y) = 1 - c_i(y) - w_i \tag{1}
\]

The expression on the right-hand side of Equation (1) is determined by the primitives of the contest. Since there are \( m \) prizes, if \( G \) is continuous at \( y \) the expression on on left-hand side equals \( \Pi_{j \in \mathbb{N} \setminus \{i\}} (1 - G_j(y)) \), i.e., the probability that all other players choose
scores higher than $y$. In fact, $G$ is continuous above 0. This follows from strictly increasing costs (Lemma 8 in Appendix C). Using this result, Equation (1) shows that if $y$ is a best response for player $i$, then the probability that all players $j \neq i$ choose a score higher than $y$ equals the value of the prize, net player $i$’s power and cost of choosing $y$. I refer to this net value as player $i$’s excess payoff at $y$, denoted

$$q_i(y) = 1 - c_i(y) - w_i = c_i(T) - c_i(y) > 0$$

Thus,

$$\Pi_{j \in N \setminus \{i\}} (1 - G_j(y)) = q_i(y) \tag{2}$$

if and only if $y$ is a best response for player $i$.

Considering scores $y$ slightly higher than $x$, I denote the set of players for whom all such scores are best responses by $A^+(x)$, and refer to it as the set of players active to the right of $x$:

$$A^+(x) = \{i \in N : \text{Equation (2) holds for all } y \in (x, z) \text{ for some } z > x\}$$

That $A^+(x)$ is well defined follows from constructibility of $G$. Let

$$\bar{x} = \sup \{z > x : A^+(z) = A^+(y) \text{ for all } y \in [x, z]\}$$

I refer to $\bar{x}$ as the first switching point above $x$, i.e., the first score higher than $x$ at which the set of players active to the right of $x$ changes. By constructibility and continuity, if $j \notin A^+(x)$, then $G_j$ does not increase on $[x, \bar{x}]$. Thus, given $G(x)$, Equation (2) for players $j \in A^+(x)$ and score $y \in (x, \bar{x}]$ leads to a system of $|A^+(x)|$ equations (where $|A|$ denotes the cardinality of a set $A$) in $|A^+(x)|$ unknowns ($1 - G_j(y)$). The unique solution for $|A^+(x)| \geq 2$ is given by Equation (3) below, and $|A^+(x)| \geq 2$ is guaranteed by strictly increasing costs (Lemma 9 in Appendix C).

**Lemma 2** Let $D = \Pi_{j \notin A^+(x)} (1 - G_j(x))$. For every $y \in [x, \bar{x}] \cap [x, T)$,

$$G_i(y) = \begin{cases} 
1 - \frac{\Pi_{j \notin A^+(x)} q_j(y) |A^+(x)|^{-1}}{q_i(y) D |A^+(x)|^{-1}} & \text{if } i \in A^+(x) \\
G_i(x) & \text{if } i \notin A^+(x)
\end{cases} \tag{3}$$

and $G_i(T) = \lim_{y \to T} G_i(y)$.

The proof of this and other results in this section is found in Appendix C. By Equation (3), the value of $G$ on $[x, \bar{x}] \subseteq [0, T]$ is determined by $A^+(x)$ and $G(x)$. $G$ can therefore be constructed on $[0, T]$ if we:
1. Determine $G(0)$.

2. For every $x \geq 0$, determine $A^+(x)$ from $G(x)$.

3. Identify $\bar{x}$, the first switching point higher than $x$.

4. Show that the number of switching points is finite.

The following lemma determines $G(0)$.

**Lemma 3** $G_i(0) = 0$ for $i < m + 1$, and $G_{m+1}(0) = \min_{i \leq m} w_i < 1$.

To determine $A^+(x)$ from $G(x)$, consider Equation (2) for a player $i \in A^+(x)$ as $y$ approaches $x$ from above. By right-continuity of $q_i$ and $G$ at $x$ (recall that costs are piecewise analytical),

$$ \Pi_{j \in N \setminus \{i\}} (1 - G_j(x)) = q_i(x) \quad (4) $$

For $x > 0$, this condition is equivalent to $x$ being a best response for player $i$ as discussed above. Because of player $m + 1$’s atom at 0, this is not so for $x = 0$ and $i \neq m + 1$. To determine $A^+(x)$, however, only Equation (4) is important. Therefore, let

$$ \forall x \geq 0 : A(x) = \{ i \in N : \text{Equation (4) holds} \} \quad (5) $$

I refer to $A(x)$ as the set of players active at $x$. Indeed, for any $x > 0$ these are the players for whom $x$ is a best response. Since Equation (4) is a necessary condition for players in $A^+(x)$, $A^+(x) \subseteq A(x)$. Since $A(x)$ is determined by $G(x)$, the inclusion $A^+(x) \subseteq A(x)$ provides a partial characterization of $A^+(x)$ given $G(x)$: players who are active to the right of $x$ must be active at $x$. The reverse, however, need not be true. That is, a player who is active at $x$ may not be active to the right of $x$. Nevertheless, $A^+(x)$ can be uniquely determined from $A(x)$. To see this, I now rewrite Equation (2) in terms of marginal percentage changes.

Recall that costs are piecewise analytical, and choose $z \in (x, \bar{x})$ such that the costs of all players in $A(x)$ are analytical on $[x, z)$. This implies that $q_i$ and $G$ are right-continuously differentiable on $[x, z)$, since $G$ is given by Equation (3). Denote by $\varepsilon_i(y) = -\frac{q_i'(y)}{q_i(y)} > 0$ player $i$’s semi-elasticity at $y$, and by $h_j(y) = -\frac{(1-G_j(y))'}{1-G_j(y)} > 0$ player $j$’s hazard rate at $y$, where all derivatives denote right-derivatives. For $i \in A^+(x)$, by Equation (2), player $i$’s semi-elasticity and hazard rate are

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19 The correspondence $x \Rightarrow A(x)$ can be thought of as “right upper-hemi continuous”. In general, however, it is not "right lower hemi-continuous", so it may be that $A^+(x)$ is a strict subset of $A(x)$. This is the case at $x_1$ in Figure 3.
excess payoff at \( y \in (x, z) \) equals the product of the other players’ probabilities of choosing scores higher than \( y \). Thus, \( \varepsilon_i (y) \) equals the sum of the other players’ hazard rates at \( y \), 
\[
\sum_{j \in N \setminus \{i\}} h_j (y).
\]
Since players who are not in \( A^+ (x) \) have hazard rates of 0 at \( y \),
\[
\forall i \in A^+ (x) : \varepsilon_i (y) = \sum_{j \in A^+ (x) \setminus \{i\}} h_j (y) \quad \text{for all } y \in (x, z)
\]
(6)

By right-continuity, Equation (6) holds at \( x \). Since, in addition, no player can obtain more than his power on a right-neighborhood of \( x \),
\[
\forall i \in A (x) : \varepsilon_i (x) \geq \sum_{j \in A^+ (x) \setminus \{i\}} h_j (x) \quad \text{with equality for } i \in A^+ (x)
\]
Letting \( H (x) = \sum_{j \in A^+ (x)} h_j (x) \),
\[
\forall i \in A (x) : h_i (x) = \max \{ H (x) - \varepsilon_i (x), 0 \}
\]
(7)

This condition, Equation (7), depends only on players’ semi-elasticities at \( x \), and can be used to determine players’ hazard rates at \( x \). To see this, think of the right-hand side of Equation (7) with \( H (x) = H \) as player \( i \)’s “supply curve” as a function of “price” \( H \). Then, \( S_x (H) = \sum_{i \in A (x)} \max \{ H - \varepsilon_i (x), 0 \} \) is the aggregate supply of “hazard rates” at \( x \) given \( H \). In equilibrium, by adding up Equation (7) for \( i \in A (x) \), the aggregate “hazard rates” supplied must equal the actual aggregate hazard rate \( H (x) \). Thus, \( H (x) \) must satisfy \( S_x (H (x)) = H (x) \). To determine \( H (x) \) from \( S_x \), note that \( S_x \) is a piecewise linear function, whose slope increases by 1 every time \( H \) exceeds the semi-elasticity of a player in \( A (x) \). Since all semi-elasticities are positive and \( |A^+ (x)| \geq 2 \), \( S'_x (0) = 0 \) and \( H (x) \neq 0 \). So, \( S_x \) is a convex function that starts below the diagonal and reaches a slope of at least 2. Therefore, it intersects the diagonal precisely once above 0, at \( H (x) \) (see Figure 4 below).

Since players with a positive hazard rate are in \( A^+ (x) \), if \( \varepsilon_i (x) < H (x) \) for player \( i \in A (x) \), then \( i \in A^+ (x) \). Since a player \( l \in A (x) \) must obtain his power immediately to the right of \( x \) to be in \( A^+ (x) \), if \( \varepsilon_i (x) > H (x) \), then \( l \notin A^+ (x) \). This is depicted in Figure 4: \( A (x) = \{ i, j, l \} \), and \( A^+ (x) = \{ i, j \} \), since \( \varepsilon_i (x) > H (x) \). Also, \( S'_x \) does not increase at \( \varepsilon_k (x) \), since player \( k \notin A (x) \).

A complication arises when \( \varepsilon_i (x) = H (x) \) for a player \( i \in A (x) \). The correct assignment of such a player is important, since his semi-elasticity slightly above \( x \) may differ from the aggregate hazard rate, so he may or may not be active to the right of \( x \). To resolve this problem, it is natural to try to apply the same fixed-point method at scores \( y \) “immediately to the right of \( x \)”, i.e., compute \( H (y) \) and see whether \( \varepsilon_i (y) \leq H (y) \). That this can be done unambiguously follows from the assumption of piecewise analytical costs.
In fact, Lemma 10 in Appendix C shows that $A^+(x)$ can be determined from $H(x)$ as follows. Compare $\varepsilon_i(x)$ and $H(x)$; if they are equal, compare their first right-derivatives, etc. (This will “generically” stop at the first derivatives.) If all derivatives are equal, $i \in A^+(x)$. The set determined by this procedure is the unique set $\overline{A^+} \subseteq A(x)$ that contains at least two players and could be used to \textbf{extend $G$ to the right of $x$ with respect to $\overline{A^+}$}, i.e., to define $G$ using Equation (3) with $A^+(x) = \overline{A^+}$ on some right-neighborhood of $x$, such that the hazard rates of all players in $\overline{A^+}$ are non-negative, and no player in $N \setminus \overline{A^+}$ can obtain more than his power.\(^{20}\)

Once $A^+(x)$ is determined, observe that for the first switching point $\bar{x}$ above $x$, i.e., the first score for which $A^+(x) \neq A^+(\bar{x})$, $j \in A^+(\bar{x}) \setminus A^+(x)$ implies that $j \in A(\bar{x}) \setminus A^+(x)$, so $j$ obtains his power at $\bar{x}$ ($P_j(\bar{x}) - c_j(\bar{x}) = w_j$). If, on the other hand, $j \in A^+(x) \setminus A^+(\bar{x})$, then $h_j(\bar{x}) \leq 0$. Thus, to identify $\bar{x}$ consider the first candidate switching point $y > x$ such that $P_j(y) - c_j(y) = w_j$ for a player $j \notin A^+(x)$, or $h_j(y) = 0$ for a player $j \in A^+(x)$,

\(^{20}\) $A^+(x)$ can also be constructed as follows. Order the players in $A(x)$ in any non-decreasing order of semi-elasticity on some right-neighborhood of $x$. $A^+(x)$ is the subset $\{1, \ldots, L(x)\} \subseteq A(x)$, where $L(x)$ is the highest $l \geq 2$ (in this ordering ) such that

$$\frac{1}{l-1} \sum_{j \in A(y), j \leq l} \varepsilon_j(y) - \varepsilon_l(y) \geq 0$$

on this right-neighborhood of $x$. This follows from solving the system of Equations (6) and using the arguments in the proof of Lemma 10.
or $y$ is a concatenation point of the cost function of a player in $A^+(x)$ (recall that costs are piecewise-defined functions). Using Equation (5), determine $A(y)$ from $G(y)$, and use $H(y)$ to determine $A^+(y)$ from $A(y)$. If $A^+(y) \neq A^+(x)$, then $\bar{x} = y$. If $A^+(y) = A^+(x)$, then $y$ is not a true switching point, and the search continues above $y$ for the next candidate switching point. This can only repeat a finite number of times before $\bar{x}$ is identified.21

Since $G(0)$ is given by Lemma 3, set $x = 0$ and begin to construct $G$ by (i) identifying $A(x)$ from $G(x)$ using Equation (5), (ii), determining $A^+(x)$ from $A(x)$ by using $H(x)$, (iii) extending $G$ to the right of $x$ with respect to $A^+(x)$, (iv) identifying the next switching point $\bar{x}$ as described above, (v) setting $x = \bar{x}$, and repeating (i)-(v) until $x = T$.

For every $x < (0,T)$ which has been reached in this process, we have:

1. $G$ is continuous and non-decreasing on $(0,x)$ by construction.
2. $1 - \Pi_{j \neq i} (1 - G_i(x)) - c_i(x) \leq w_i$, with equality if $h_i(x) > 0$ by construction.
3. $G(x) < 1$. This follows from the continuity of $G$ up to $x$, since $G(0) < 1$ (Lemma 3), and if $G_i(x) \geq 1$, then every player $j \neq i$ would obtain strictly more than his power by playing slightly lower than $x$.
4. $|A(x)| \geq 2$. This can be seen by induction on the number of switching points up to $x$, since $A(0) = \{i \in N : w_i \leq \min_{j < m+1} w_j\}$ so $|A(0)| \geq 2$.

Points 3 and 4 show that the construction can be continued from any score $x < T$ that has been reached. The following lemma shows that the number of switching points that result is finite.

**Lemma 4** The number of switching points in $[0,T]$ that result from the construction is finite. Moreover, $A(x) = N$ for all $x$ sufficiently close to $T$.

The construction will therefore reach $T$ by applying (i)-(v) above a finite number of times. Thus, $G$ is characterized by a partition into a finite number of intervals of positive length, on the interior of which the set of active players remains constant. The value of $G$ on each interval is given by Equation (3). This completes the description of the algorithm. To show that $G$ is an equilibrium, it remains to show that $G_i(T) = 1$.

**Lemma 5** For every player $i$, $G_i(T) = 1$.

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21This can be shown using analyticity, as in the proof of Lemma 10. Players $j \in A^+(x)$ for whom $\varepsilon_i^{(k)}(x) = H^{(k)}(x)$ for all $k \geq 0$ are ignored in the search for the first candidate switching point, etc.
Proposition 3 \( G \) is a constructible equilibrium.

Proof. \( G \) is a profile of distribution functions, since it is right-continuous on \([0, T]\), weakly increasing, and \( G(T) = 1 \) (point 1 above and Lemma 5). By point 2 above, no player can obtain more than his power, and \( G_i \) is strictly increasing only where player \( i \) obtains precisely his power. Thus, best responses are chosen with probability 1, so \( G \) is an equilibrium. By the construction procedure, \( G \) is constructible. \( \blacksquare \)

To gain more intuition for the construction procedure, consider how the supply function \( S_x \) and its positive fixed point \( H(x) \) work in the context of Figure 3 above. \( A(0) = A^+(0) = \{2, 3\} \). As \( x \) increases from 0 to \( T \), the set of active players changes. At the switching point \( x_1 \), player 1 becomes active since he obtains his power. This changes \( S_x \) and \( H(x) \) discontinuously. As a result, \( H(x_1) \) falls below player 2’s hazard rate, and he becomes inactive above \( x_1 \). At \( x_2 \), player 2 rejoins the set of active players, and all three players are active up to \( x_3 \). Thus, the addition of an active player may or may not cause another to become inactive. At \( x_3 \), player 3’s hazard rate reaches 0, and he becomes inactive above \( x_3 \). He rejoins the set of active players at \( x_4 \), and all three players remain active up to the threshold.\(^{22}\)

6.2 Equilibrium Uniqueness and Implications

Since the value of \( G \) at 0, the switching points, and the corresponding sets of active players are uniquely determined, we have the following.

Corollary 3 \( G \) is the unique constructible equilibrium of an \( m + 1 \)-player regular contest.

Proof. Consider a constructible equilibrium \( \tilde{G} \), and denote by \( \tilde{x} \) the supremum of the scores on which \( \tilde{G} \) coincides with \( G \). Since \( \tilde{G}(0) = G(0) \) (Lemma 3), and both \( \tilde{G} \) and \( G \) are continuous above 0 (costs are strictly increasing), \( \tilde{G}(\tilde{x}) = G(\tilde{x}) \). By constructibility of \( \tilde{G} \) and the construction of \( G \), \( \tilde{G}(y) = G(y) \) on a right-neighborhood of \( \tilde{x} \), for \( \tilde{x} < T \). Thus, \( \tilde{x} = T \) (and \( G(T) = \tilde{G}(T) = 1 \)). \( \blacksquare \)

Corollary 3 does not apply to non-constructible equilibria, since it assumes that for every \( x \in [0, T] \) there is a right-neighborhood on which the set of active players remains

\(^{22}\)Bulow & Levin (2005) constructed the equilibrium of a game in which players have linear costs and compete for heterogeneous prizes. Their construction proceeds from the top, without first identifying players’ equilibrium payoffs. This is possible because each player’s best-response set is an interval. Such a procedure does not work here, as the set of players active to the \textit{left} of \( x \) is not uniquely determined by \( G(x) \).
constant. To show that the output of the algorithm is the unique equilibrium, the existence of equilibria that are not constructible must be ruled out. This is done in Proposition 4.\textsuperscript{23}

**Proposition 4** If an \(m+1\)-player contest with strictly increasing costs has a constructible equilibrium \(G\), then \(G\) is the unique equilibrium of the contest.

Combining Proposition 4 and Lemma 4, we have the following.

**Theorem 3** An \(m+1\)-player regular contest has a unique equilibrium, which is constructible. It is characterized by a partition of \([0,T]\) into a finite number of closed intervals with disjoint interiors of positive length, such that the set of active players is constant on the interior of each interval. Thus, each player’s best-response set is a finite union of intervals. All players are active on the last interval.

**Proof.** Immediate. ■

Equilibrium uniqueness shows that equivalent players play identical strategies.

**Corollary 4** In an \(m+1\) regular contest, if \(\frac{c_i(\cdot)}{v_i} = \frac{c_j(\cdot)}{v_j}\) on \([0,T]\), then \(G_i = G_j\), where \(G\) is the unique equilibrium of the contest.

**Proof.** If \(G_i\) were different from \(G_j\), then switching them would lead to a second equilibrium, contradicting uniqueness. ■

I now turn to regular contests with any number of players.

**Corollary 5** Regular contests have at most one equilibrium in which \(m+1\) players participate. The candidate for this equilibrium is the unique equilibrium of the reduced contest with players \(1, \ldots, m+1\). It is an equilibrium of the original contest if and only if players \(m+2, \ldots, n\) cannot obtain positive payoffs by participating. If they can, then in every equilibrium at least \(m+2\) players participate.

**Proof.** Since costs are strictly increasing, players \(1, \ldots, m+1\) participate in every equilibrium. Thus, an equilibrium in which precisely \(m+1\) players participate must coincide with the unique equilibrium of the reduced contest that involves players \(1, \ldots, m+1\). ■

In some regular contests, only players \(1, \ldots, m+1\) participate in any equilibrium. A sufficient condition for this is that the normalized cost functions of players with negative

\textsuperscript{23}To the best of my reading, Baye et al. (1996), Clark & Riis (1998), and Bulow & Levin (2005), all of whom constructed equilibria of similar games with a continuum of pure strategies in which more than two players participate, did not rule out the existence of equilibria that are not constructible.
power are strictly higher than that of the marginal player (Proposition 2). By Corollary 5, such contests have a unique equilibrium, given by the algorithm.

Even when the equilibrium is unique, the algorithm shows that gaps in a player’s best-response set may occur when his semi-elasticity of excess payoff, which is closely related to his semi-elasticity of costs, is sufficiently higher than those of other active players. Thus, such gaps derive from local cost disadvantages. In the special case where costs are linear transformations of a common function, players’ semi-elasticities are identical. No player is locally disadvantaged, so every player’s best-response set is an interval as the following proposition shows.

**Proposition 5** A regular contest in which \( c_i (\cdot) = \gamma_i c (\cdot) \) for every player \( i \) has a unique equilibrium, in which the best-response set of every player \( j \leq m + 1 \) is an interval of the form \([\beta_j, T]\). Moreover, for \( i, j \leq m \), \( \beta_i \geq \beta_j \) if and only if \( \frac{w_i}{v_i} \geq \frac{w_j}{v_j} \). \( \beta_{m+1} = 0 \). Players \( i > m + 1 \) do not participate, i.e., choose score 0 with certainty.

The intuition is that since players’ semi-elasticities are identical, the set of players active at \( x \) equals the set of players active to the right of \( x \), for every \( x \in [0, T) \). In fact, all players \( i \) active at \( x \) have the same hazard rate, since \( h_i (x) = H (x) - \varepsilon_i (x) = \frac{H(x)}{|A(x)|} > 0 \). Thus, once a player becomes active, he remains active up to the threshold. The higher the power of a player, the later he becomes active.

Setting \( c (x) = x \), Proposition 5 shows that a generic all-pay auction has a unique equilibrium, in which every active player \( i \) plays on an interval \([\beta_i, T]\). The higher player \( i \)'s valuation, the higher is \( \beta_i \). \(^{24}\)

### 6.3 Two-Player Contests

Two-player, one-prize contests are simpler to solve because equilibrium strategies exist such that both players are always active. This is shown in Proposition 6 below. Since the set of active players does not change, the regularity conditions are not necessary. In fact, an equilibrium can be constructed regardless of the Generic Conditions.

**Proposition 6** In any two-player, one-prize contest with initial scores of 0, generic or not, \((G_1, G_2) = \left( \frac{c_2(\cdot)}{v_2}, \frac{w_1 + c_1(\cdot)}{v_1} \right)\) on \([0, T]\) is an equilibrium in which player payoffs equal

\(^{24}\)This result generalizes that of Clark & Riis (1998). They constructed an equilibrium for multi-prize all-pay auctions in which different players have different valuations, but gave an incorrect argument for its uniqueness.
their powers. If costs are strictly increasing (regardless of the Power Condition), this is the unique equilibrium of the contest.

Proposition 6 extends the results of Kaplan & Wettstein (2006) and Che & Gale (2006), who considered two-player contests with strictly increasing, ordered cost functions. When costs are not strictly increasing, a generic contest may have other equilibria (see Example 4). When the Cost Condition is not met, equilibria may exist in which a player obtains more than his power (see Example 1). An immediate consequence of Proposition 6 and the Threshold Lemma is the following.

**Corollary 6** A generic one-prize contest with initial scores of 0 and strictly increasing costs has at most one equilibrium in which two players participate. When Proposition 2 holds, the equilibrium is unique.

**Proof.** Similar to that of Corollary 5. ■

## 7 Conclusion

Contests capture sunk investments and general asymmetries among contenders. I have provided a closed-form formula for players’ expected payoffs in generic contests, and used it to compute aggregate expenditures, derive comparative statics, and analyze player participation. I then described an algorithm that constructs the unique equilibrium for a subclass of contests, which nests generic all-pay auctions.

The analysis of contests illustrates which predictions of the all-pay auction for \( m \) prizes depend on the linear cost assumption:

1. For each of the strongest \( m \) players, linearity of his payoff in the difference between his valuation and that of the marginal player. This occurs because a player’s power equals this difference, and the strongest \( m \) players obtain a payoff equal to their power (Theorem 1). When costs are not linear, powers, and therefore payoffs, may not be linear in the difference in players’ valuations.

2. Complete rent dissipation as valuations approach a common value. This follows from prediction 1, using the aggregate expenditures formula of Section 4. The same formula shows that in other contests rent dissipation may be as low as zero even when valuations are identical.
3. Participation by precisely $m+1$ players. This derives from the assumption of ordered costs, as shown in Proposition 2. When costs are not ordered, more than $m+1$ players may participate (Theorem 2).

4. Equilibrium uniqueness. This follows from Corollary 5 of Theorem 3, since all-pay auctions are regular contests in which precisely $m+1$ players participate. Other regular contests may have multiple equilibria (Example 3).

5. Interval best-response sets. This happens when players’ costs are a linear transformation of a common function, as shown by Proposition 5. In other regular contests with a unique equilibrium, a player’s best response set may be a union of several disjoint intervals (the example of Section 6).

The comparison between contests and all-pay auctions shows that asymmetries among contenders may significantly influence the outcome of a competition, and suggests that we may be able to empirically distinguish between linear and non-linear costs. The payoff and equilibrium construction results have implications for optimal contest design. I hope to pursue these issues in further research.
References


A Examples 1-4

Example 1 The Cost Condition and the payoff characterization do not hold.
Let \( n = 2, m = 1, a_i = 0, v_i = 1, c_1 (x) = bx \) for \( b < 1 \), and
\[
c_2 (x) = \begin{cases} 
\frac{x}{d} & \text{if } 0 \leq x < d \\
1 & \text{if } d \leq x \leq 1 \\
2x - 1 & \text{if } x > 1 
\end{cases}
\]
for \( d < 1 \). Then \( r_1 = \frac{1}{b} > 1, T = r_2 = 1, w_1 = 1 - b, w_2 = 0 \) and the Power Condition holds. Cost Condition fails, however, because \( c_2^{-1} (c_2 (r_2)) = [d, 1] \). As a result, \( G = (G_1, G_2) \) is an equilibrium in which player 1 gets a payoff of \( 1 - bd > w_1 \), for
\[
G_1 (x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x}{d} & \text{if } 0 \leq x \leq d \\
1 & \text{if } x > d 
\end{cases}, G_2 (x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 - bd + bx & \text{if } 0 \leq x \leq d \\
1 & \text{if } x > d 
\end{cases}
\]
As \( b \) approaches 1, player 1’s power approaches 0; for any value of \( b \), as \( d \) approaches 0 player 1’s payoff approaches 1, the value of the prize. The Zero Lemma, which does not require the Generic Conditions, shows that Player 2 has a payoff of 0.

Example 2 The Power Condition and the payoff characterization do not hold.
Let \( n = 3, m = 1, a_i = 0, v_i = 0 \), and
\[
c_1 (x) = \begin{cases} 
(2 - 2\alpha) x & \text{if } 0 \leq x \leq \frac{1}{2} \\
(1 - \alpha) + (x - \frac{1}{2}) 2\alpha & \text{if } x > \frac{1}{2} 
\end{cases}, c_2 (x) = \begin{cases} 
(2 - 2\varepsilon) x & \text{if } 0 \leq x \leq \frac{1}{2} \\
(1 - \varepsilon) + (x - \frac{1}{2}) 2\varepsilon & \text{if } x > \frac{1}{2} 
\end{cases}, c_3 (x) = \begin{cases} 
\gamma x & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{\gamma}{2} + (x - \frac{1}{2}) 2L & \text{if } x > \frac{1}{2} 
\end{cases}
\]
for small \( \alpha, \varepsilon \in (0, 1) \), small \( \gamma \geq 0 \), and some \( L > 0 \). This is a version of the example of Section 1.1.

Costs are strictly increasing at 1 for all players, so the Cost Condition is met. Without re-indexing players, \( r_1 = r_2 = 1 \) and \( r_3 = \frac{1+L}{2L} \) so for \( L > 1 \), \( r_3 < 1 \) and for \( L < 1 \), \( r_3 > 1 \). Regardless of the value of \( L \), the Power Condition is violated (since two players have the same reach and, therefore, the same power), player 2 is the marginal player, and the threshold is 1.
I now show that for any $L > 0$ there exists $\beta > 0$ such that for $\alpha, \varepsilon, \gamma < \beta$, $(G_1, G_2, G_3)$ is an equilibrium, for

$$G_1(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\gamma \frac{x - \frac{1}{2}}{1 - \alpha} + (1 - \varepsilon) & \text{if } 0 \leq x \leq \frac{1}{2}
\end{cases}$$

$$G_2(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 - \alpha & \text{if } 0 \leq x \leq \frac{1}{2}
\end{cases}$$

$$G_3(x) = \begin{cases} 
2x & \text{if } x < \frac{1}{2} \\
1 & \text{if } x \geq \frac{1}{2}
\end{cases}$$

Players’ costs and the equilibrium $(G_1, G_2, G_3)$ are depicted in Figure 4 below.

![Figure 4](image.png)

**Figure 4:** Cost functions and the equilibrium $(G_1, G_2, G_3)$ of Example 2

Fix $L > 0$. For $\alpha, \varepsilon \leq \frac{1}{10}$, $G$ is a profile of CDFs, since $G_1(0) \geq 0$ and $G$ is non-decreasing in $x$. To show that $G$ is an equilibrium, I show that (a) player 1 obtains a payoff of 0 on $[0, 1)$, (b) player 2 obtains a payoff of 0 on $\{0\} \cup [\frac{1}{2}, 1)$ and $\leq 0$ on $(0, \frac{1}{2})$, and (c) player 3 obtains a payoff of $(1 - \varepsilon) (1 - \alpha) - \frac{\gamma}{2} > 0$ on $(0, \frac{1}{2})$ and a weakly lower payoff on $\{0\} \cup (\frac{1}{2}, 1)$, for small enough $\varepsilon$ and $\alpha$.

I do this by considering three cases: (i) $x = 0$, (ii) $x \in (0, \frac{1}{2}]$, and (iii) $x \in (\frac{1}{2}, 1)$.
Case (i): Because \( G(0) = \left((1 - \varepsilon) - \frac{\gamma}{2(1 - \alpha)}, 1 - \alpha, 0\right) \), \( u_1(0) = u_2(0) = 0 \) and \( u_3(0) \leq (1 - \varepsilon)(1 - \alpha) - \frac{\gamma}{2} \).

Case (ii):

1. Player 1 obtains a payoff of 0:

\[
u_1(x) = G_2(x)G_3(x) - c_1(x) = (1 - \alpha)2x - (2 - 2\alpha)x = 0
\]

2. Player 2 obtains a payoff \( \leq 0 \): since \( u_2(0) = u_2\left(\frac{1}{2}\right) = 0 \), and \( f(x) = G_1(x)G_3(x) - c_2(x) \) is differentiable, it suffices to prove that it has exactly one extremum on \((0, \frac{1}{2})\), and that this extremum is a minimum. I show that \( f' \) has one root in \((0, \frac{1}{2})\), and that \( f'' \) is positive at that root:

\[
f'(x) = G'_1(x)G_3(x) + G_1(x)G'_3(x) - c'_2(x) = \frac{\gamma}{1 - \alpha}G_3(x) + 2G_1(x) - (2 - 2\varepsilon) = \frac{\gamma}{1 - \alpha}2x + \frac{2\gamma}{1 - \alpha}x - \frac{\gamma}{1 - \alpha} + 2(1 - \varepsilon) - (2 - 2\varepsilon) = \frac{\gamma}{1 - \alpha}(4x - 1)
\]

so \( f'(x) = 0 \iff x = \frac{1}{4} \), and \( f''(x) = \frac{4\gamma}{1 - \alpha} > 0 \).

3. Player 3 obtains a payoff of \((1 - \varepsilon)(1 - \alpha) - \frac{\gamma}{2}\):

\[
u_3(x) = G_1(x)G_2(x) - c_3(x) = \left(\gamma \frac{x - \frac{1}{2}}{1 - \alpha} + (1 - \varepsilon)\right)(1 - \alpha) - \gamma x = \\
\gamma \left(x - \frac{1}{2}\right) + (1 - \varepsilon)(1 - \alpha) - \gamma x = (1 - \varepsilon)(1 - \alpha) - \frac{\gamma}{2}
\]

Case (iii):

1. Player 1 obtains a payoff of 0:

\[
u_1(x) = G_2(x)G_3(x) - c_1(x) = (1 - \alpha) + \left(x - \frac{1}{2}\right)2\alpha - (1 - \alpha) + \left(x - \frac{1}{2}\right)2\alpha = 0
\]

2. Player 2 obtains a payoff of 0:

\[
u_2(x) = G_1(x)G_3(x) - c_2(x) = (1 - \varepsilon) + \left(x - \frac{1}{2}\right)2\varepsilon - (1 - \varepsilon) + \left(x - \frac{1}{2}\right)2\varepsilon = 0
\]

3. Player 3 obtains a payoff \( < (1 - \varepsilon)(1 - \alpha) - \frac{\gamma}{2} \): let \( x = \frac{1}{2} + \delta \) for \( \delta \in (0, \frac{1}{2}) \). Since

\[
u_3(x) = G_1(x)G_2(x) - c_1(x) = ((1 - \varepsilon) + 2\delta\varepsilon)((1 - \alpha) + 2\delta\alpha) - 2\delta L - \frac{\gamma}{2}
\]
it suffices to show that

\[(1 - \varepsilon) + 2\delta \varepsilon) \left( (1 - \alpha) + 2\delta \alpha \right) - 2\delta L - \frac{\gamma}{2} < (1 - \varepsilon) (1 - \alpha) - \frac{\gamma}{2} \]

or, equivalently, that

\[\alpha + \varepsilon - 2\alpha\varepsilon + 2\delta \alpha \varepsilon < L\]

for which a sufficient condition is \(\alpha, \varepsilon < \frac{L}{2}\).

Finally, \((1 - \varepsilon) (1 - \alpha) - \frac{\gamma}{2} > 0\) for \(\gamma < 2 (1 - \varepsilon) (1 - \alpha)\). Thus, \((G_1, G_2, G_3)\) is an equilibrium for \(\alpha, \varepsilon, \gamma < \beta = \min \left\{ \frac{1}{16}, \frac{L}{2} \right\}\).

Player 3 can be made either the strongest player (by setting \(L < 1\)) or the weakest player \((L > 1)\). In the former case, it is a player in \(N_L \setminus \{m + 1\}\) who has power 0. In the latter case, it is a player in \(N_W\) who has power 0. Regardless of the value of \(L\), as \(\varepsilon, \alpha, \) and \(\gamma\) tend to 0 player 3 wins with near certainty, and his payoff approaches the value of the prize. Increasing \(L\), player 3 can have arbitrarily low power.\(^{25}\)

Note that:

1. If we changed \(v_1\) or \(v_2\) slightly, Theorem 1 would hold and player 3’s payoff would equal his power.

2. Costs are strictly increasing for \(\gamma > 0\).

3. Setting \(\gamma = 0\), \((G_1, G_2, 1_{x \geq \frac{1}{2}})\) is an equilibrium in which player 3’s payoff tends to 1 as \(\alpha\) and \(\varepsilon\) tend to 0.

**Example 3**  Strictly increasing costs and two equilibria, in which different players participate and in which aggregate expenditures differ.

Let \(n = 4\) and \(m = 1\). I extend the example of Section 1.1 by adding player 4, without constructing the contest explicitly. I will demonstrate the existence of two equilibria, \(G\) and \(G'\), such that only players 1,2, and 3 participate in \(G\), and only players 1,2, and 4 participate in \(G'\). Different player participation implies different allocations, since a player chooses a costly score only if he has a positive probability of winning by doing so.

Begin with an equilibrium \(G = (G_1, G_2, G_3)\) of the example of Section 1.1. When \(\gamma\) is small and positive, all three players participate in \(G\) as shown in Section 4.1. Let \(t_3 = \inf \{x : G_3(x) = 1\}\). I will add player 4 with \(v_4 = 1\) and continuous, strictly increasing

\(^{25}\)The Generalized Threshold Lemma shows that \(N_T = \{1, 2\}\). The Zero Lemma identifies two players with payoff 0, players 1 and 2. Thus, player 3’s payoff is not constrained.
costs $c_4$ that are lower than those of player 3 below $t_3$ and equal to them starting from $t_3$. That is, $\forall x \in (0, t_3) : c_4(x) \in (0, c_3(x))$, and $\forall x \geq t_3 : c_4(x) = c_3(x)$. Note that $c_4 \neq c_3$.

I now show that there exist such functions $c_4$, for which an equilibrium is for players 1, 2 and 3 to play $G$, and for player 4 not to participate. Indeed, assume that player 4 does not participate. Since player 3’s equilibrium payoff is 0 (his power is negative), he obtains at most 0 by choosing any score when players 1 and 2 play $G_1$ and $G_2$, and he wins a prize under $G$ if and only if he beats players 1 and 2. By Lemma 8 in Appendix C, there are no atoms above 0 so $\forall x > 0 : P_3(x) = G_1(x) G_2(x)$. Player 4, who considers joining in when the others are playing $G$, must beat players 1, 2, and 3 to win a prize, i.e., $\forall x > 0 : P_4(x) = G_1(x) G_2(x) G_3(x)$. Let $x \in (0, t_3)$, which implies that $G_3(x) < 1$. If $u_3(x) = 0$, then $P_3(x) > 0$ because $c_3(x) > 0$. Thus, $P_4(x) < P_3(x)$ and $P_4(x) - c_3(x) < P_3(x) - c_3(x) = u_3(x) = 0$. If $u_3(x) < 0$, since $P_4(x) \leq P_3(x)$ again $P_4(x) - c_3(x) < 0$. Thus, there exist continuous, non-decreasing functions $c_4$ such that $\forall x \in (0, t_3) : c_4(x) \in (0, c_3(x))$ and $P_4(x) - c_4(x) < 0$. For such functions, it is a best response for player 4 not to participate. Since $G$ is an equilibrium of the contest that includes only players 1, 2, and 3, we have an equilibrium.

Now maintaining the same cost functions, consider an equilibrium $G' = (G'_1, G'_2, G'_4)$ of the contest that includes only players 1, 2, and 4. As in the example of Section 1.1, all three players must participate in $G'$. Adding player 3 leads to an equilibrium when player 3 doesn’t participate because player 4’s payoff is zero (his power is negative), and at every score player 3’s costs are weakly higher than those of player 4 whereas his probability of winning is weakly lower than that of player 4.

If aggregate expenditures under the two equilibria are the same, multiply the valuation and cost of player 4 by some positive $d \neq 1$. This does not change the equilibria of the contest (Lemma 7 in Appendix B), but changes aggregate expenditures in $G'$.

**Example 4** Multiple equilibria when costs are not strictly increasing.

Let $n = 2, m = 1, a_i = 0, v_i = v$,

$$
c_1(x) = \begin{cases} 
0 & \text{if } x < X \\
 x - X & \text{if } x \geq X 
\end{cases}
$$

and $c_2(x) = x$. This is a version of an all-pay auction, in which player 1 has an initial advantage of $X$. For $X \geq v$, the only equilibrium is a pure strategy equilibrium: player 1 chooses $X$, and player 2 chooses 0. Player 1 wins with certainty, and expenditures are zero.
For $X < v$, $(F_1, F_2)$ is an equilibrium, where

\[
F_1(x) = \begin{cases} 
H_1(x) & \text{if } x < X \\
\frac{x}{v} + \frac{(x-X)}{v} & \text{if } X \leq x \leq v \\
1 & \text{if } x > v 
\end{cases}
\]

and any right-continuous, non-decreasing function $H_1$, such that $H_1(0) = 0$ and $H_1(x) \leq \frac{x}{v}$. Setting $a_1 = X$ would rule out these equilibria for $H_1 \neq 0$.

### B Proofs of the Results of Sections 2, 3, and 4

The following notation is used in the proofs. Recall that a mixed-strategy of player $i$ is a cumulative probability distribution $G_i$. When a strategy profile $G = (G_1, \ldots, G_n)$ is specified, denote by $BR_i$ the set of player $i$’s best responses. $P_i(\cdot)$ is expanded to mixed strategies, $P_i(G_i')$ serves as shorthand for $P_i(G_{-i}, G_i')$, $P_i(x)$ is the equivalent notation when player $i$ chooses $x \geq a_i$ with certainty, and similarly for $u_i(\cdot)$. For an equilibrium $(G_1, \ldots, G_n)$, denote by $u_i = u_i(G_i)$ player $i$’s equilibrium payoff. Note that in equilibrium best responses are chosen with probability 1.

The phrase “player $i$ beats player $j$” refers to player $i$ choosing a strictly higher score than player $j$. “Winning” means obtaining one of the $m$ prizes. “Losing” means not obtaining a prize. For a set $I$, denote by $|I|$ the cardinality of $I$.

#### B.1 Proof of Theorem 1

Fix an equilibrium $G = (G_1, \ldots, G_N)$, and recall that $\forall i \in N_W : w_i \geq 0$ and $\forall i \in N_L : w_i \leq 0$.

**Proof of Lemma 1.** Each player can guarantee himself a payoff of 0 by choosing his initial score $a_i$. Thus, it suffices to consider players with positive power, all of whom (there may be none) are in $N_W$.

Since any player $i$ can guarantee himself a payoff of 0 by choosing $a_i$ and $\forall i \in N_L : T \geq r_i$, it must be that $\forall i \in N_L : G_i(T) = 1$. Thus, $\forall i \in N_W, \forall \varepsilon > 0 : P_i(\max \{T + \varepsilon, a_i\}) = 1$ since by choosing $\max \{T + \varepsilon, a_i\}$ player $i$ beats all players in $N_L$ with certainty. Hence, $\forall i \in N_W : u_i \geq v_i - c_i (\max \{T + \varepsilon, a_i\}) \to w_i$ by continuity of $c_i$.

**Proof of the Tie Lemma.** Denote by $N'$ the set of players who have an atom at $x$,
and let \( n' = |N'| \geq 2 \). Denote by \( E \) the positive-probability event that all players in \( N' \) choose \( x \). Denote by \( D \subseteq E \) the event in which a relevant tie occurs at \( x \), i.e., the event in which \( m' \leq m \) prizes are divided among the \( n' \) players in \( N' \) for \( 1 \leq m' < n' \). Suppose \( D \) has positive probability. Then, conditional on \( D \), at least one player in \( N' \) can strictly increase his probability of winning to 1 by choosing a score slightly higher than \( x \). Therefore, \( D \) has probability 0. This implies that \( E = E^L \cup E^W \), where \( E^L \) is the event that at least \( m \) players in \( N \setminus N' \) choose scores \( > x \), and \( E^W \) is the event that at most \( m - n' \) players in \( N \setminus N' \) choose scores \( > x \). By independence of players’ strategies, either \( E^L \) or \( E^W \) has probability 0 otherwise \( D \) would have positive probability. Thus, \( E = E^L \) or \( E = E^W \). Suppose that \( E = E^L \). Independence of players’ strategies now implies that without conditioning on \( E \) at least \( m \) players in \( N \setminus N' \) choose scores \( > x \) with probability 1, so we have \( P_i(x) = 0 \) for every \( i \in N' \). Similarly, if \( E = E^W \) then \( P_i(x) = 1 \) for every \( i \in N' \). ■

**Proof of the Zero Lemma.** Assume that the lemma does not hold, and denote by \( J \) a set of \( m + 1 \) players who do not have best responses with which they win with arbitrarily small probability. Let \( \tilde{S} = \cup_{i \in J} BR_i \) and \( s_{\inf} = \inf(\tilde{S}) \). There are three cases to consider: (1) two or more players in \( J \) have an atom at \( s_{\inf} \), (2) exactly one player in \( J \) has an atom at \( s_{\inf} \), and (3) no player in \( J \) has an atom at \( s_{\inf} \).

In case (1), denote by \( N' \subseteq J \) the set of players in \( J \) who have an atom at \( s_{\inf} \). It cannot be that \( P_i(s_{\inf}) = 1 \) for all players \( i \in N' \): even if all \( n - m - 1 \) players in \( N \setminus J \) choose scores strictly lower than \( s_{\inf} \) with probability 1, only

\[
m - |(J \setminus N')| = m - (m + 1 - |N'|) = |N'| - 1
\]

prizes are divided among the \( |N'| \) players in \( N' \). Thus, the Tie Lemma shows that \( P_i(s_{\inf}) = 0 \) for all \( i \in N' \).

In case (2), denote by \( i \) the only player in \( J \) with an atom at \( s_{\inf} \). Then, \( P_i(s_{\inf}) = 0 \) since all \( m \) players in \( J \setminus \{i\} \) choose scores strictly higher than \( s_{\inf} \) with probability 1.

In cases (1) and (2), \( P_i(s_{\inf}) = 0 \) for some player \( i \in J \) who has an atom at \( s_{\inf} \), which is therefore a best response for him.

In case (3), by definition of \( s_{\inf} \) there exists a player \( i \) in \( J \) with best responses \( \{x_n\}_{n=1}^\infty \) that approach \( s_{\inf} \). Since \( 1 \geq 1 - P_i(x_n) \geq \prod_{j \in J \setminus \{i\}} (1 - G_j(x_n)) \) and no player has an atom at \( s_{\inf} \), as \( n \) tends to infinity \( P_i(x_n) \) approaches 0. This completes the proof. ■

I prove a more general version of the Threshold Lemma, the Generalized Threshold Lemma, which relaxes the Generic Conditions. For the statement of the lemma, the following condition is required.
**Condition 1** If player $i$ has power 0, then $c_i(x) < c_i(T)$ for every $x < T$.

When only the marginal player has power 0, this condition coincides with the Cost Condition specified in the Generic Conditions.

**Generalized Threshold Lemma** Denote by $N_T$ the set of players who play at least up to the threshold, i.e., $N_T = \{i \in N : \forall x < T : G_i(x) < 1\}$. If Condition 1 holds, then:

1. $|N_T| \geq m$.
2. $\forall i \in N_T : w_i \geq 0$.
3. If $|N_T| = m$ then $\forall i \in N_T : c_i(x) = c_i(T)$ on some left-neighborhood of $T$.

The proof of the Generalized Threshold Lemma uses the following corollary of the Zero Lemma.

**Corollary 7** There exists a player with non-negative power who has a payoff of 0.

**Proof.** Every player with negative power belongs to $N_L \setminus \{m + 1\}$. Thus, there are at most $N - m - 1$ players with negative power, and the Zero Lemma shows that at least $N - m$ players have a payoff of 0. ■

Denote this player by $i_0$.

**Proof of the Generalized Threshold Lemma.** Let $t_i = \inf \{x : G_i(x) = 1\}, J = \{i : t_i < T\}$, and $s_{\text{max}} = \max_{i \in J} t_i$. Obviously, $s_{\text{max}} < T$. Suppose that part (1) of the lemma is false. Then $J \geq n - m + 1$, and by choosing $s_{\text{max}} + \frac{T - s_{\text{max}}}{2}$ player $i_0$ beats all players in $J \setminus \{i_0\}$, so $P_{i_0}(s_{\text{max}} + \frac{T - s_{\text{max}}}{2}) = 1$. Also, $c_i(s_{\text{max}} + \frac{T - s_{\text{max}}}{2}) \leq c_i(T)$. Thus, $w_{i_0} > 0$ implies that $u_{i_0}(s_{\text{max}} + \frac{T - s_{\text{max}}}{2}) \geq w_i > 0$, contradicting the previous corollary. If $w_{i_0} = 0$, by Condition 1 $c_{i_0}(s_{\text{max}} + \frac{T - s_{\text{max}}}{2}) < c_{i_0}(T)$ so again $u_{i_0}(s_{\text{max}} + \frac{T - s_{\text{max}}}{2}) > 0$.

For part (2), note that a player $j$ with negative power obtains a negative payoff by choosing scores sufficiently close to the threshold, so $G_j(T - \varepsilon) = 1$ for some $\varepsilon > 0$.

For part (3), if $|N_T| = m$ then $\forall i \in N_T : P_i(s_{\text{max}} + \frac{T - s_{\text{max}}}{2}) = 1$ since by choosing $s_{\text{max}} + \frac{T - s_{\text{max}}}{2}$ player $i$ beats all $n - m$ players in $J$. If the claim is false, then at least one player in $N_T$ would be better off by choosing a score slightly lower than the threshold and still winning with certainty, a contradiction. ■

**Corollary 8** The payoff of players in $N_T$ equals at most their power.
Proof. Let \( i \in N_T \). If \( \max \{ T, a_i \} = a_i \), then \( w_i = v_i \geq u_i \). Suppose \( \max \{ T, a_i \} = T > a_i \). Since \( \forall x < T : G_i(x) < 1, \exists \{ x_n \}_{n=1}^{\infty} : x_n \to z \geq T, x_n \in BR_i \). Thus,
\[
u_i = u_i(x_n) = P_i(x_n) v_i - c_i(x_n) \leq v_i - c_i(x_n) \to v_i - c_i(z) \leq v_i - c_i(T) = w_i\]

Lemma 6  If the Generic Conditions hold, then \( N_W \subseteq N_T \).

Proof. The Generic Conditions imply that the Generalized Threshold Lemma holds. Since every player in \( N_T \) has non-negative power by part two of the Generalized Threshold Lemma, \( N_T \subseteq N_W \cup \{ m + 1 \} \). \(|N_T| \geq m \) and \(|N_W| = m \), so \( N_W \not\subseteq N_T \) implies \( \{ m + 1 \} \subseteq N_T \) and \(|N_T| = m \), which contradicts the last part of the Generalized Threshold Lemma.

The lemma and the previous corollary show that under the Generic Conditions the payoff of players in \( N_W \) equals at most their power. Theorem 1 follows, as in Section 3.

B.2  Proof of Proposition 1

The proof requires the following lemma.

Lemma 7  Multiplying the valuation and cost function of a player by a constant \( \gamma > 0 \) does not change the set of equilibria of a contest.

Proof. Since a player’s von Neumann-Morgenstern preferences are invariant to increasing affine transformations of his utility function, the new contest is equivalent to the original one.

By taking \( \gamma = \frac{1}{v_i} \), it suffices to prove the proposition for contests in which valuations equal 1. Consider such a contest \( C \) and the restricted game \( C' \), in which \( S_i = [a_i, K] \), for \( K = \max_{i \in N} r_i \) (\( r_i \) is defined in Definition 2). Any equilibrium of \( C' \) is an equilibrium of \( C \), since scores higher than \( K \) are strictly dominated by \( a_i \) for every player \( i \). Thus, it suffices to show that \( C' \) has an equilibrium.

The pure-strategy game \( C' \) is compact and Hausdorff, and players’ utilities are bounded. I will apply Proposition 5.1 and Corollary 5.2 of Reny (1999). It is immediate that \( \sum_{i=1}^{n} u_i(s) = m - \sum_{i=1}^{n} c_i(s_i) \) is upper semi-continuous. For payoff security, consider player \( i \), a strategy profile \( G = (G_1, \ldots, G_n) \), and a score \( s_i \) such that \( u_i(s_i, G_{-i}) \geq u_i(G) \). If \( s_i = K \), then \( u_i(G) \leq 0 \) by definition of \( K \) so player \( i \) can guarantee himself 0 by
choosing $a_i$.

If $s_i < K$, pick a score $s_i' > s_i$ such that $c_i(s_i') - c_i(s_i) < \frac{\varepsilon}{2}$ for $\varepsilon > 0$. Let
\[
n_{\delta_i} (G_{-i}) = \times_{j \neq i} \{ \text{probability measures } \tilde{G}_j : \int f d\tilde{G}_j < \alpha_j + \delta \}\]
for $\delta > 0$, $\alpha_j = \int f dG_j$, and
\[
f(x) = \begin{cases} 
0 & \text{if } x \leq s_i \\
\frac{x - s_i}{s_i' - s_i} & \text{if } s_i < x < s_i' \\
1 & \text{if } x \geq s_i'
\end{cases}
\]

It is a technical exercise to show that
\[
\exists \delta > 0 \text{ such that } \forall \tilde{G}_{-i} \in n_{\delta} (G_{-i}) : P_i (s_i', \tilde{G}_{-i}) \geq P_i (s_i, G_{-i}) - \frac{\varepsilon}{2}
\]

**B.3 Proof of Theorem 2**

In the construction that follows, all players have strictly increasing costs and initial scores of 0. Lemma 8 in Appendix C shows that in equilibrium there are no atoms above 0, and Lemma 9 in Appendix C shows that every score in $(0, T)$ is a best response for at least two players.

The proof proceeds in three stages: (1) for any $n > 1$, I construct a generic contest with $n$ players and one prize, in which all players participate, (2) for any $m > 1$, I extend this contest by adding $m - 1$ prizes and $m - 1$ players, such that all players participate, and (3) I add any number of players who do not participate. The value of the prize is set to 1; distinct valuations are achieved by multiplying a player’s valuations and cost function by any positive constant. Lemma 7 shows that this does not change the set of equilibria of the contest.

**Stage 1.** Fix $n \geq 2$. For $n = 2$, let
\[
c_i(x) = \begin{cases} 
4x_\alpha (1 - \gamma) & \text{if } x < \frac{1}{4} \\
\alpha (1 - \gamma) + 2\alpha \gamma \left( x - \frac{1}{4} \right) & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\
\alpha + 8 \left( x - \frac{3}{4} \right) (\beta) & \text{if } \frac{3}{4} \leq x \leq \frac{7}{8} \\
\alpha + \beta + 8i \left( x - \frac{7}{8} \right) (1 - \alpha - \beta) & \text{if } x > \frac{7}{8}
\end{cases}
\]
for $i \in \{1, 2\}$, $\alpha \in (0, 1)$ close to 1, and small $\beta, \gamma > 0$ such that $\alpha + \beta < 1$. The Threshold Lemma shows that both players choose scores up to $r_2 = T > \frac{7}{8}$, so both players...
participate.

For \( n > 2 \), let \( K = n - 2 \) and partition \([\frac{1}{4}, \frac{3}{4}]\) to \( 2K \) segments \( L_t, t \in [1 \ldots 2K] \) of length \( \frac{1}{4K} \), so that \( L_t = [\frac{3}{4} - \frac{t}{4K}, \frac{3}{4} - \frac{t-1}{4K}] = [L_t^L, L_t^R] \). Define \( c_1(\cdot) \) and \( c_2(\cdot) \) as in the case \( n = 2 \). For \( k \in 3, \ldots, n \), define \( c_k(\cdot) \) in the following way, using \( c_1(\cdot), \ldots, c_{k-1}(\cdot) \). Let \( q_k = \min_{t \leq k-1} c_t \left( L_{2(k-2)}^b \right) \in (0, 1) \), and

\[
c_k(x) = \begin{cases} 
4x (q_k^3 - \gamma^{nk}) & \text{if } x < \frac{1}{4} \\
(q_k^3 - \gamma^{nk}) + \frac{\gamma^{nk}}{L_{2(k-2)}^R - \frac{1}{4}} \left( x - \frac{1}{4} \right) & \text{if } \frac{1}{4} \leq x \leq L_{2(k-2)}^L \\
q_k^3 + 4K \left( x - L_{2(k-2)}^R \right) (1 - q_k^3) & \text{if } x > L_{2(k-2)}^L = L_{2(k-2)}^R 
\end{cases}
\]

for \( \gamma^{nk} \) such that \((q_k^3 - \gamma^{nk}) > 0\). Note that \( r_k = L_{2(k-2)}^L \), so the contest is generic, \( T = r_2 \), and players \( 3, \ldots, n \) have negative powers. Since costs are strictly increasing, the Threshold Lemma shows that players 1 and 2 choose scores up to the threshold and therefore participate in every equilibrium.

I now show that players \( 3, \ldots, n \) must also participate in every equilibrium. Suppose, in contradiction, that there is an equilibrium in which player \( k > 2 \) does not participate. This equilibrium is clearly an equilibrium of the reduced contest without player \( k \). Consider this reduced contest and the corresponding reduced equilibrium \( \bar{G} \). Since the reach of players \( k + 1, \ldots, n \) is at most \( L_{2(k-2)}^b \), they have no best responses above \( L_{2(k-2)}^L \) and therefore choose scores above \( L_{2(k-2)}^b \) with probability 0. Recall that every score in \((0, T)\) is a best response for at least two players, and let \( i_x, j_x < k \) be two players for whom the score \( x \) in the interior of \( L_{2(k-2)} \) is a best response. Since players have non-negative equilibrium payoffs, it must be that \( P_{i_x}(x) = \Pi_{r < k, r \neq i_x} \bar{G}_r(x) \geq q_k \) and similarly for \( j_x \). Thus, \( \Pi_{r < k} \bar{G}_r(x) \geq q_k^3 \) for every score in the interior of \( L_{2(k-2)} \). Since \( \forall x \in L_{2(k-2)} : c_k(x) \leq q_k^3 < \Pi_{r < k} \bar{G}_r(x) \), player \( k \) can obtain a positive payoff under \( \bar{G} \) in the original contest by choosing a score in the interior of \( L_{2(k-2)} \). Since his power is negative, this cannot be an equilibrium.

This proves that every player participates in every equilibrium of the \( n \)-player, single-prize contest constructed. Denote this contest by \( C \) and its threshold by \( T_c \).

**Stage 2.** Recall that \( r_i \leq \frac{3}{4} \) for \( i \geq 3 \) and in equilibrium every score in \((0, T_c)\) is a best response for at least two players. Thus, in any equilibrium \( G \),

\[
G_1 \left( \frac{7}{8} \right) = c_2 \left( \frac{7}{8} \right), G_2 \left( \frac{7}{8} \right) = c_1 \left( \frac{7}{8} \right) + u_1 \tag{8}
\]

Since players 1 and 2 choose scores up to the threshold \( T_c > \frac{7}{8} \), \( G_1 \left( \frac{7}{8} \right), G_2 \left( \frac{7}{8} \right) < 1 \). Fix an \( \varepsilon \in \left( \max \left\{ G_1 \left( \frac{7}{8} \right), G_2 \left( \frac{7}{8} \right) \right\}, 1 \right) \), and note that \( \varepsilon \) is independent of the specific equilibrium \( G \). Now add \( m - 1 \) prizes and \( m - 1 \) players \( i \) with \( a_i = 0 \), \( c_i(\cdot) \) strictly increasing, and \( c_i(T_c) < (1 - \varepsilon)^2 \). The threshold of the new \( m \)-prize, \( n + m - 1 \)-player
contest is $T_c$, the powers of the $n$ original players do not change, and those of the new players exceed $1 - (1 - \varepsilon)^2$. For the remainder of the proof “player 1” and subscript 1 refer to player 1 of the contest $C$, and similarly “player 2” and subscript 2.

Fix an equilibrium $\tilde{G}$ of the new contest. Since costs are strictly increasing, the Threshold Lemma shows that players 1 and 2 participate, as do the new players. To show that the $n - 2$ remaining players also participate, I first show that $\tilde{G}_i \left( \frac{3}{4} \right) = 0$ for every new player $i$. For this, it suffices to demonstrate that that $\tilde{s_{inf}}$, the infimum of the union of the best-response sets of the new players, is $> \frac{7}{8}$. Indeed, $\tilde{s_{inf}} > 0$ otherwise one of the new players would have a payoff of 0, as in the Zero Lemma, but $1 - (1 - \varepsilon)^2 > 0$. Since $\tilde{s_{inf}} > 0$ and there are no atoms above 0, by definition of $\tilde{s_{inf}}$ $u_j (\tilde{s_{inf}}) = P_j (\tilde{s_{inf}}) - c_j (\tilde{s_{inf}})$ for some new player $j$. Also, $P_j (\tilde{s_{inf}}) \leq 1 - \left( 1 - \tilde{G}_1 (\tilde{s_{inf}}) \right) \left( 1 - \tilde{G}_2 (\tilde{s_{inf}}) \right)$: if player $j$ wins a prize, it must be that player 1, player 2, or both chose a score lower than $\tilde{s_{inf}}$, since all other new players choose scores higher than $\tilde{s_{inf}}$ with probability 1. Since $c_1 \left( \frac{3}{4} \right) + u_1 < \varepsilon$ (Equation (8) and the definition of $\varepsilon$), it must be that $\tilde{G}_2 \left( \frac{7}{8} \right) < \varepsilon$, otherwise player 1 could obtain in $\tilde{G}$ more than his power by choosing $\frac{7}{8}$. Similarly $\tilde{G}_1 \left( \frac{7}{8} \right) < \varepsilon$. Thus, if $\tilde{s_{inf}} \leq \frac{7}{8}$ then

$$1 - (1 - \varepsilon)^2 < u_j (\tilde{s_{inf}}) < P_j (\tilde{s_{inf}}) \leq 1 - \left( 1 - \tilde{G}_1 (\tilde{s_{inf}}) \right) \left( 1 - \tilde{G}_2 (\tilde{s_{inf}}) \right) \leq 1 - (1 - \varepsilon)^2$$

a contradiction. Consequently, $\tilde{s_{inf}} > \frac{7}{8}$, and $\tilde{G}_i \left( \frac{3}{4} \right) = 0$ for every new player $i$.

Therefore, on $[0, \frac{3}{4}]$ only the $n$ original players compete for one prize. The participation arguments from stage (2) applied to the restriction of $\tilde{G}$ to $[0, \frac{3}{4}]$ show that the remaining $n - 2$ players also participate in $\tilde{G}$.

Stage 3. By Proposition 2, any additional player with costs strictly higher than those of the marginal player does not participate in any equilibrium.

### B.4 Proof of Proposition 2

By Lemma 7 it suffices to prove the result for a contest in which valuations equal 1. Choose an equilibrium $G$ of such a contest, and suppose player $i > m + 1$ with costs strictly dominated by those of the marginal player invested positively in $G$. Let $t_i = \inf \{ x : G_i (x) = 1 \}$. Then $c_i (t_i) > 0$ and $u_i (t_i) \geq 0$. By assumption, $c_i (t_i) - c_{m+1} (\max \{ a_{m+1}, t_i \} ) = \varepsilon > 0$. For every $\delta > 0 : P_{m+1} (\max \{ a_{m+1}, t_i \} + \delta) \geq P_i (t_i)$ since player $m + 1$ beats player $i$ for sure and beats the other players at least as often as player $i$ does. By continuity of $c_{m+1}$, player $m + 1$ can choose $\max \{ a_{m+1}, t_i \} + \delta$ for some small $\delta > 0$, such that $c_i (t_i) - c_{m+1} (\max \{ a_{m+1}, t_i \} + \delta) \geq \frac{\varepsilon}{2} > 0$. Thus, $u_{m+1} \geq P_{m+1} (\max \{ a_{m+1}, t_i \} + \delta) - c_{m+1} (\max \{ a_{m+1}, t_i \} + \delta) > P_i (t_i) - c_i (t_i) = u_i (t_i) \geq 0$, or $u_{m+1} > 0$, which contradicts $w_{m+1} = 0 = u_{m+1}$.
C Proofs of the Results of Section 6

Lemma 8 Any equilibrium of a contest with strictly increasing costs and initial scores 0 is continuous above 0.

Proof. Suppose a player $i$ had an atom at $x > 0$. Then, similarly to the proof of the Tie Lemma, another player $j$ would choose a score $y$ slightly below $x$ only if he wins with probability 0 or 1 when choosing $y$. The former cannot happen since $j$ would be better off choosing 0. The latter cannot happen since $i$ could win with probability 1 by choosing any score in $(y, x)$. Thus, no player $j \neq i$ chooses scores in some region below $x$. Therefore, player $i$ would be better off by choosing a score slightly below $x$. This shows that $G$ is continuous above 0.

Lemma 9 In a contest with strictly increasing costs and initial scores of 0, every score in $(0, T]$ is a best response of at least two players.

Proof. If $x < T$ is not a best response for player $i$, continuity of $G$ implies that the same is true for scores in some neighborhood of $x$. Thus, if only one player had a best response at $x$, he could choose scores slightly lower than $x$ and win with the same probability, making him better off. If no player has a best response at $x$, then there is a gap in the union of players’ best response sets. Since costs are strictly increasing, continuity of $G$ implies that no player would choose scores at the top of the gap, contradicting the Threshold Lemma.

C.1 Proof of Lemma 2

Choose $y \in (x, \bar{x})$, and let $r_i(y) = 1 - G_i(y)$. Since $q_i(y) > 0$ and $r_i(y), D > 0$ (all players choose scores up to the threshold by the Threshold Lemma and strictly increasing costs), Equation (2) for $i \in A^+(x)$ can be rewritten as $\Pi_{j \in A^+(x) \backslash \{i\}} r_j(y) = \frac{q_i(y)}{D} > 0$. Taking natural logs,

$$\sum_{j \in A^+(x) \backslash \{i\}} \ln r_j(y) = \ln q_i(y) - \ln D$$

This is a system of $|A^+(x)|$ linear equations in $|A^+(x)|$ unknowns $r_j(y)$. Denote by $I_{M \times M}$ and $1_{M \times M}$ the identity matrix and a matrix of ones, respectively, of dimensions $M \times M$. Then, in vector notation,

$$\left(1_{|A^+(x)| \times |A^+(x)|} - I_{|A^+(x)| \times |A^+(x)|}\right) \ln r(y) = \ln q(y) - \ln D$$

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Since \((1_{M \times M} - I_{M \times M})^{-1} = \left(\frac{1}{M} \cdot 1_{M \times M} - I_{M \times M}\right)\), we have

\[
\ln r_i(y) = \frac{1}{|A^+(x)| - 1} \sum_{j \in A^+(x)} \ln q_j(y) - \ln q_i(y) - \frac{1}{|A^+(x)| - 1} \ln D
\]

which gives the result for \(y \in (x, \bar{x})\). For \(y \in \{x, \bar{x}\}\), the result follows from left-continuity at \(x\) and right-continuity above \(0\).

C.2 Proof of Lemma 3

Since positive payoffs imply winning with positive probability at every best response, the Tie Lemma shows that players in \(1, \ldots, m\) do not have an atom at \(0\). Since strictly increasing costs imply that there are no atoms above \(0\) and every \(x > 0\) is a best response of at least two players (Lemma 9), \(G_{m+1}(0) \geq \min_{i \leq m} w_i\). Since no player should be able to obtain more than his power by choosing a score slightly above \(0\), \(G_{m+1}(0) \leq \min_{i \leq m} w_i\). Strictly increasing costs also imply that all positive powers (and payoffs) are strictly smaller than \(1\), so \(G_{m+1}(0) < 1\).

C.3 Statement and Proof of Lemma 10

I show that \(A^+(x)\) is exactly all players \(i \in A(x)\) with \(\varepsilon_i(y) \leq H^{A(x)}(y)\) on some right-neighborhood of \(x\), where \(H^{A(x)}(y)\) is the positive fixed point of

\[
S_y(H|A(x)) = \sum_{j \in A(x)} \max \{H - \varepsilon_k(y), 0\}
\]

Then, by point 1 below, to determine whether \(\varepsilon_i(y) \leq H^{A(x)}(y)\) on some right-neighborhood of \(x\), it suffices to compare the derivatives of \(\varepsilon_i(x)\) and \(H(x)\), as specified in Section 6, since \(S_x(\cdot|A(x)) = S_x(\cdot)\) so \(H^{A(x)}(x) = H(x)\).

**Lemma 10** A set \(\overline{A^+} \subseteq A(x)\) satisfies the following three conditions:

1. \(\overline{A^+} \geq 2\)

2. For every player \(i \in \overline{A^+}\), \(h_i(y) \geq 0\) on some right-neighborhood of \(x\)

3. For every player \(i \notin \overline{A^+}\), \(1 - \Pi_{j \neq i}(1 - G_j(y)) - c_i(y) < w_i\) on some right-neighborhood of \(x\)
when $G$ is extended to the right of $x$ with respect to $\overline{A^+}$, if and only if $\overline{A^+} = \widehat{A}^+ (x)$ for

$$\widehat{A}^+ (x) = \{ i \in A(x) : \varepsilon_i (y) \leq H^A (x) (y) \text{ on some right-neighborhood of } x \}$$

The following points assist in the proof:

1. $\exists \hat{x}$ such that $\forall i \in A(x), \forall A^+ \subseteq A(x), |A^+| \geq 2 : H^{\overline{A^+}} - \varepsilon_i$ is analytical on $[x, \hat{x}]$.

   Indeed, since costs are analytical on $[x, \hat{x}]$ for some $\hat{x} > x$ so are semi-elasticities. From Equation (6), $H^{\overline{A^+}} (y) = \frac{1}{|A|} \sum_{j \in A^+} \varepsilon_j (y)$ so $H^{\overline{A^+}} (y)$ on $[x, \hat{x}]$ is analytical as well. Since an analytical function with an accumulation point of roots is identically 0 in the connected component of the accumulation point, $\exists \delta_i > 0$ such that $\forall A^+ \subseteq A(x), |A^+| \geq 2 : H^{\overline{A^+}} (y) - \varepsilon_i (y)$ is strictly positive, strictly negative, or identically 0 for $y \in (x, x + \delta_i)$, $x + \delta_i < T$. Let $\delta = \min_{i \in A(x)} \delta_i$.

2. $\widehat{A}^+ (x) = \{ i \in A(x) : \varepsilon_i (y) \leq H^{\widehat{A}^+} (x) (y) \text{ for } y \in [x, x + \delta) \}$, since $H^{\widehat{A}^+} (x) (y) = H^A (x) (y)$ for $y \in [x, x + \delta)$, by the definition of $\widehat{A}^+ (x)$, $S_y (\cdot |A^+) (x)$, and $S_y (\cdot |A(x))$.

3. Condition 2 in the lemma is met for $\overline{A^+}$ if and only if when $G$ is extended to the right of $x$ with respect to $\overline{A^+}$ the aggregate hazard rate is $H^{\overline{A^+}}$. This follows from uniqueness of the positive fixed point of $S_y (\cdot |A^+)$. 

\textbf{Proof.} \textit{“if”}: That condition 1 is satisfied is immediate from the definition of $H^A (x) (y)$. Now for every $i \in \widehat{A}^+ (x)$, let $\hat{h}_i (y) = H^{\widehat{A}^+} (x) (y) - \varepsilon_i (y) \geq 0$ for $y \in [x, x + \delta)$, where the inequality follows from point 2 above. Then $\forall i \in \widehat{A}^+ (x)$ Equation (6) is met, so player $i$’s hazard rate is $\hat{h}_i (y) \geq 0$, and the aggregate hazard rate is $H^{\widehat{A}^+} (x) (y)$, for $y \in [x, x + \delta)$. Condition 3 is met for players in $A(x) \setminus \widehat{A}^+ (x)$ since, by point 2, such players have semi-elasticities on $(x, \delta)$ strictly higher than the aggregate hazard rate $H^{\widehat{A}^+} (x)$. Condition 3 is trivially met for players in $N \setminus A(x)$ by continuity of $G$.

\textit{“only if”}: Consider a set $\overline{A^+} \subseteq A(x), |\overline{A^+}| \geq 2$, that satisfies conditions 2 and 3. Since $\overline{A^+} \subseteq A(x), S_y (\cdot |\overline{A^+}) \leq S_y (\cdot |A(x))$, so $H^{\overline{A^+}} (x) = H^A (x) \leq H^{\overline{A^+}}$. This implies that $\widehat{A}^+ (x) \subseteq \overline{A^+}$. Otherwise, when $G$ is extended to the right of $x$ with respect to $\overline{A^+}$ every player in $\widehat{A}^+ (x) \setminus \overline{A^+}$ obtains at least his power on $(x, x + \delta)$, since by point 3 above the aggregate hazard rate is $H^{\overline{A^+}}$, which violates condition 3. Since $\widehat{A}^+ (x) \subseteq \overline{A^+}, H^{\overline{A^+}} \leq H^{\widehat{A}^+}$. By condition 2 and point 3, when $G$ is extended to the right of $x$ with respect to $\overline{A^+}, \forall i \in \overline{A^+} : \varepsilon_i (y) \leq H^{\overline{A^+}} (y)$ for $y \in [x, x + \delta)$. So, by point 2, $\overline{A^+} \subseteq \widehat{A}^+ (x)$. Therefore, $\overline{A^+} = \widehat{A}^+ (x)$. 

$A^+ (x)$ is the set specified by the lemma, since it satisfies conditions 1-3 when $G$ is extended to the right of $x$ with respect to $A^+ (x)$.
C.4 Proof of Lemma 4

Lemma 11 \( \forall \tilde{x} < T, \) the number of switching points in \([0, \tilde{x}]\) is finite.

Proof. I assume analytical cost functions (the obvious extension applies to piecewise-analytical functions). Fix \( \tilde{x} < T \) and rank players’ semi-elasticities at every score in \([0, \tilde{x}]\). Since semi-elasticities are analytical, this ranking can change only finitely many times. Thus, \([0, \tilde{x}]\) can be divided into a finite number of intervals such that the ranking of players’ semi-elasticities is constant on each interval. Fix one such interval \( I \). For every subset \( B \subseteq N \) of at least two players and every \( x \in J \), denote by \( t_B (x) \) the highest semi-elasticity of a player who can join the set of active players \( B \) and maintain a non-negative hazard rate: 
\[
t_B (x) = \frac{1}{|B| - 1} \sum_{j \in B} \varepsilon_j (x) \quad \text{(the aggregate hazard rates of players in } B)\].

Since semi-elasticities are analytical, so is \( t_B (\cdot) \). Thus, the interval \( J \) can be divided into a finite number of subintervals such on every subinterval each function in \( \{ \varepsilon_i - t_B : i \in N, B \subseteq N, |B| \geq 2 \} \) is either positive, negative, or 0. Clearly, on any such subinterval \( L \subseteq J \) an active player can become inactive only if a player with a strictly lower semi-elasticity becomes active (recall that the order of players’ semi-elasticities doesn’t change on \( J \)). Now, suppose in contradiction that the number of switching points in \( L \) is infinite. This implies that some player \( i \) becomes inactive and active an infinite number of times, which, by the above, implies that some player \( j \) with semi-elasticity strictly lower than that of \( i \) becomes inactive and active an infinite number of times. Continuing in this way, we obtain a contradiction since the number of players is finite. ■

The following lemmas show that there are no switching points on some left-neighborhood of \( T \).

Lemma 12 \( \exists \tilde{x} < T \) such that \( \forall i \in N : \varepsilon_i (x) < H (x) \) for every \( x \in (\tilde{x}, T) \).

Proof. First, \( \forall i, j : \)
\[
\lim_{x \to T} \frac{\varepsilon_i (x)}{\varepsilon_j (x)} = \frac{c_i' (x) c_j (T) - c_j' (x) c_i (x)}{c_j' (x) c_i (T) - c_i' (x) c_j (x)} = \lim_{x \to T} \frac{c_j (T) - c_j (x)}{c_i (T) - c_i (x)} = \lim_{x \to T} \frac{c_j' (T)}{c_j (T) - c_j (x)} \frac{(c_j (T) - c_j (x))'}{c_i (T) - c_i (x)} = 1
\]
where the penultimate equality follows from l’Hospital’s rule.

Let \( \varepsilon_{\min} (x) = \min_{i \in N} \varepsilon_i (x) \) for \( x < T \). Then, by the above, 
\[
\lim_{x \to T} \frac{\varepsilon_i (x)}{\varepsilon_{\min} (x)} = 1, \quad \text{so } \frac{\varepsilon_i (x)}{\varepsilon_{\min} (x)} < \frac{n}{n - 1} \text{ for all } x > \tilde{x} \text{ for some } \tilde{x} \text{ sufficiently close to } T.
\]
To conclude, it suffices to show that \( \forall x > \tilde{x} : H (x) \geq \frac{n}{n - 1} \varepsilon_{\min} (x) \). Let \( S_x^{\min} (H) = n \max \{ H - \varepsilon_{\min} (x), 0 \} \). Then \( \forall H : S_x (H) \leq S_x^{\min} (H) \) and since \( \frac{n}{n - 1} \varepsilon_{\min} (x) \) is the unique positive fixed point of \( S_x^{\min} \), \( H (x) \geq \frac{n}{n - 1} \varepsilon_{\min} (x) \). ■

Since active players with semi-elasticities strictly lower than the aggregate hazard rate remain active, in order to complete the proof it suffices to show the following.
Lemma 13 Every player i has scores x arbitrarily close to T such that \( q_i(x) = \Pi_{j \neq i} (1 - G_j(x)) \).

Proof. Suppose, in contradiction, that \( \forall x \in (\tilde{x}_i, T): q_i(x) < \Pi_{j \neq i} (1 - G_j(x)) \) for some player i and some \( \tilde{x}_i > \tilde{x} \) of the previous lemma. Then, \( f(x) = \sum_{j \neq i} \ln (1 - G_j(x)) - \ln q_i(x) > 0 \). Since \( i \notin A(x) \),

\[
\forall x \in (\tilde{x}_i, T) : H(x) = \frac{|A^+(x)|}{|A^+(x)| - 1} \sum_{j \in A^+(x)} \varepsilon_j(x) = \sum_{j \in N} \frac{G_j'(x)}{(1 - G_j(x))} = \sum_{j \neq i} \frac{G_j'(x)}{(1 - G_i(x))}
\]

Thus,

\[
f'(x) = \varepsilon_i(x) - \frac{|A^+(x)|}{|A^+(x)| - 1} \sum_{j \in A^+(x)} \varepsilon_j(x) \leq \varepsilon_i(x) - \frac{n - 1}{n - 2} \sum_{j \in A^+(x)} \varepsilon_j(x) =
\]

\[-\frac{1}{n - 2} \varepsilon \min(x) + o(\varepsilon \min(x)) \]

as \( x \to T \), by the proof of the previous lemma. Since

\[-\frac{1}{n - 2} \int_x^T \varepsilon \min(y) dy = \lim_{z \to T} \frac{1}{n - 2} (\ln q \min(z) - \ln q \min(x)) = -\infty \]

\( f \) crosses 0 at a score in \( (\tilde{x}_i, T) \), a contradiction. \( \blacksquare \)

C.5 Proof of Lemma 5

By Lemma 4, \( \exists \tilde{x} < T \) such that \( \forall x \in (\tilde{x}, T) : A(x) = N \). Equation (3) now implies that

\[
\forall x \in (\tilde{x}, T), \forall i \in N : \ln (1 - G_i(x)) = \frac{1}{n - 1} \sum_{j \neq i} \ln q_j(x) - \ln q_i(x)
\]

To show that \( G_i(x) \to 1 \) \( x \to T \), it therefore suffices to show that

\[
\frac{1}{n - 1} \sum_{j \neq i} \ln q_j(x) - \ln q_i(x) \to -\infty
\]

Since \( \ln q_i(x) \to -\infty \), it suffices to show that

\[
\frac{1}{n - 1} \sum_{j \neq i} \ln q_j(x) \ln q_i(x) > 1 + \delta \text{ for some } \delta > 0
\]

for large enough \( x \). The inequality follows from l’Hospital’s rule and the fact that \( \lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_j(x)} = 1 \), shown in the proof of Lemma 4.
C.6 Proof of Proposition 4

For expositional simplicity, I assume that the number of switching points under $G$ is finite. It is straightforward to accommodate a countably infinite number of switching points by defining the limit of a sequence of switching points to be a switching point and modifying the proof appropriately.

In the following propositions, $x_k$ denotes switching point $k$ in $G$. The last switching point is $T$. $A(x)$ and $A^+(x)$ are as defined in Section 6. Choose any equilibrium $\tilde{G}$ of the contest, and recall that $\tilde{G}$ is continuous above 0 because costs are strictly increasing. $\tilde{A}(x)$ denotes the set of players active at $x$ under $\tilde{G}$, i.e., the set of players defined by Equations (4) and (5) with $\tilde{G}$ instead of $G$. Using the fact that $A^+(x)$ exists, I show that $A(x) = \tilde{A}(x)$ for all $x \in [0, T]$. The following lemma shows that this is sufficient.

Lemma 14 Let $\bar{x} \in [0, T]$. If $\forall x \in [0, \bar{x}]: \tilde{A}(x) = A(x)$, then $\forall x \in [0, \bar{x}]: \tilde{G}(x) = G(x)$.

Proof. Similar to that of Proposition 3, since $\tilde{G}(0) = G(0)$ (Lemma 3 does not rely on analyticity), $\tilde{G}$ satisfies the conditions in the definition of constructibility on $[0, \bar{x})$ (because $G$ does), and $\tilde{G}$ is continuous above 0.

Now, let $x_k$ be the highest positive switching point such that $\tilde{A}(x) = A(x)$ on $[0, x_k]$, and suppose in contradiction that $x_k < T$. Choose $x \in (x_k, x_{k+1})$ such that $\tilde{A}(x) \neq A(x)$. Since $x_k < T$, such an $x$ exists otherwise Lemma 14 and continuity would imply that $\tilde{A}(x_{k+1}) = A(x_{k+1})$. The following lemmas show that $\tilde{A}(x) \subseteq A(x)$ and $A(x) \subseteq \tilde{A}(x)$, which completes the proof.

Lemma 15 $\tilde{A}(x) \subseteq A(x)$.

Proof. Suppose $\tilde{A}(x) \not\subseteq A(x)$, and let $i_0 \in \tilde{A}(x) \setminus A(x)$. Since $i_0 \not\in A(x)$, we have

$$c_{i_0}(x) + w_{i_0} > P_{i_0}(x) = 1 - \Pi_{j \neq i_0}(1 - G_j(x))$$

or

$$1 - (c_{i_0}(x) + w_{i_0}) < \Pi_{j \neq i_0}(1 - G_j(x))$$

and since $i_0 \in \tilde{A}(x)$, we have

$$1 - (c_{i_0}(x) + w_{i_0}) = \Pi_{j \neq i_0}\left(1 - \tilde{G}_j(x)\right)$$

so

$$\Pi_{j \neq i_0}\left(1 - \tilde{G}_j(x)\right) < \Pi_{j \neq i_0}(1 - G_j(x))$$

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Let $J_1 = N \setminus \{i_0\}$. Then

$$\Pi_{j \in J_1} \left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in J_1} \left(1 - G_j(x)\right)$$

(9)

By the Threshold Lemma, the expression on each side of Inequality (9) is a product of $n-1$ strictly positive numbers. Therefore, there exists a player $i_1 \in J_1$ such that

$$\Pi_{J_1 \setminus \{i_1\}} \left(1 - \tilde{G}_j(x)\right) < \Pi_{J_1 \setminus \{i_1\}} \left(1 - G_j(x)\right)$$

(10)

(otherwise multiplying the products of all subsets of size $n-2$ for $G$ and for $\tilde{G}$ would lead to a contradiction).

Now, note that $\forall i \in N : \tilde{G}_i(x_k) = G_i(x_k)$, by Lemma 14, and since $\tilde{G}_i$ is non-decreasing

$$\forall i \notin A^+(x) : \left(1 - \tilde{G}_i(x)\right) \leq \left(1 - G_i(x)\right)$$

(11)

Let $K_1 = N \setminus J_1 = \{i_0\}$. Since $A^+(x) \subseteq A(x)$ and $i_0 \notin A(x)$, by Inequality (11)

$$\left(1 - \tilde{G}_j_{\in K_1}(x)\right) \leq \left(1 - G_{j \in K_1}(x)\right)$$

(12)

By Inequalities (10) and (12),

$$\Pi_{j \in J_1 \cup K_1 \setminus \{i_1\}} \left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in J_1 \cup K_1 \setminus \{i_1\}} \left(1 - G_j(x)\right)$$

(13)

Since $N = J_1 \cup K_1$, Inequality (13) shows that $i_1 \notin A^+(x_k)$, otherwise $i_1$ would obtain under $\tilde{G}$ more than his power by choosing $x$.

Now repeat the process above, letting $J_{r+1} = J_r \setminus \{i_r\}, K_{r+1} = K_r \cup \{i_r\}$. By induction on $r$, Inequalities (9), (10), (12), and (13) hold with $J_1 = J_r, K_1 = K_r$, and $i_1 = i_k$. A contradiction is reached at stage $n$, since $J_n = \phi$ but $A^+(x_k) \neq \phi$.

**Corollary 9** $\forall j \notin A(x), \forall y \in (x_k, x_{k+1}) : \tilde{G}_j(y) = G_j(y) = G(x_k)$.

**Proof.** Immediate from $\tilde{A}(x) \subseteq A(x)$.

The next two lemmas establish that $A(x) \subseteq \tilde{A}(x)$.

**Lemma 16** If $A(x) \not\subseteq \tilde{A}(x)$, then $\tilde{G}_i(x) > G_i(x)$ for some $i \in A(x) \setminus \tilde{A}(x)$.

**Proof.** Let $B = A(x) \setminus \tilde{A}(x)$, and suppose that $\forall j \in B : \tilde{G}_j(x) \leq G_j(x)$. This implies $\exists j \in B : \tilde{G}_j(x) < G_j(x)$. Otherwise, by the corollary above and Equation (3) with $A^+(x) = \tilde{A}(x)$, we obtain $\tilde{G}(x) = G(x)$ and therefore $A(x) = \tilde{A}(x)$.

To show that $\exists i \in B$ such that $\tilde{G}_i(x) > G_i(x)$, the following observation is required. Fix some values $\tilde{G}_j(x)$ for $j \notin A(x)$ and use Equation (3) to solve for the values $\tilde{G}_l(x), l \in$
Now, consider the following process by which \( G(x) \) is “reached” from \( G(x) \). Let \( \tilde{G}(x) \) be the solution that \( x \) would obtain under \( G(x) \) and solve for \( A(x) \). Continuing in this way and noting that, by Equation (3) with \( A^+(x) = \tilde{A}(x) \), \( G(x) \) and \( \tilde{G}(x) \) have the same value, \( \forall x \in A(x) \cup B = N/\tilde{A}(x) \), we see that \( \tilde{G}(x) > G(x) \) for some player \( i \in B \).

Lemma 17 \( A(x) \subseteq \tilde{A}(x) \).

**Proof.** Suppose \( A(x) \not\subseteq \tilde{A}(x) \). Then, by the previous lemma, \( \tilde{G}(x) > G(x) \) for some \( i \in A(x) \setminus \tilde{A}(x) \). Since \( \tilde{G}(x) = G(x) \), \( \tilde{G}(y) > G(y) \) for some \( y \in (x_k, x) \) such that \( i \in \tilde{A}(y) \). This means that \( \tilde{A}(y) \neq A(y) \) (otherwise Equation (3) would imply \( \tilde{G}(y) = G(y) \)). Let \( \tilde{B} = A(y) \setminus \tilde{A}(y) \).

For players \( \tilde{B} = A(y) \setminus \tilde{A}(y) \), perform a procedure similar to the one above, reaching \( \tilde{G}(y) \) from \( G(y) \). Begin with players \( l \in \tilde{B} \) for whom \( \tilde{G}(y) > G(y) \). Raising \( G(y) \) to \( \tilde{G}(y) \) decreases the solutions for all other players, so the order of raising does not matter - the solutions must be raised for all players \( l \in \tilde{B} \) for whom \( \tilde{G}(y) > G(y) \). If the solutions of any remaining players in \( \tilde{B} \) now need to be raised to reach their level in \( \tilde{G} \), repeat this process until no more players in \( \tilde{B} \) need their solutions raised. It cannot be that \( \tilde{B} \) is exhausted, since \( \tilde{G}(y) > G(y) \) and so far the solutions of all players in \( \tilde{A}(y) \) have been repeatedly decreased, starting from their level in \( G \). Thus, there remains a set non-empty set \( \tilde{B} \subseteq \tilde{B} \) of players whose solutions must now be decreased to reach their level in \( \tilde{G} \). Decreasing solutions increases the solutions for all other players. Thus, by the argument used in the previous lemma, the last player whose solution is decreased receives too high a payoff under \( \tilde{G} \) at \( y \).
C.7 Proof of Proposition 5

Since \( c_i(\cdot) = \gamma_i c(\cdot) \), and only player \( m+1 \) has power 0, players \( m+2, \ldots, n \)'s normalized costs are strictly higher than those of player \( m+1 \). Thus, Proposition 2 applies, and the contest has a unique equilibrium, given by the algorithm.

For the second part of the proposition, it suffices to consider \( m+1 \)-player contests in which valuations equal 1. Given such a contest, the threshold equals \( c^{-1} \left( \frac{1}{\gamma_{m+1}} \right) \) and player \( i \)'s power equals \( 1 - c_i \left( c^{-1} \left( \frac{1}{\gamma_{m+1}} \right) \right) = 1 - \frac{\gamma_i}{\gamma_{m+1}} \). Therefore, player \( i \)'s semi-elasticity at \( x \) is independent of \( i \):

\[
- \frac{q_i'(x)}{q_i(x)} = \frac{c_i'(x)}{1 - w_i - c_i(x)} = \frac{\gamma_i c'(x)}{1 - \left( 1 - \frac{\gamma_i}{\gamma_{m+1}} \right) - \gamma_i c(x)} = \frac{\gamma_i c'(x)}{\frac{\gamma_{m+1}}{\gamma_{m+1}} - \gamma_i c(x)} = \frac{c'(x)}{\gamma_{m+1} - c(x)}
\]

Since all players have identical semi-elasticities, all players active at \( x \) are active to the right of \( x \), by definition of \( H(x) \). Consequently, once a player becomes active at a score \( x \) he remains active up to the threshold.

For the third part of the proposition, assume that \( \beta_i \leq \beta_j \) for \( i, j < m+1 \). Since \( i \) and \( j \) are not active below \( \beta_i \), this implies that \( G_i(\beta_i) = G_j(\beta_i) = 0 \). Thus, \( P_i(\beta_i) = P_j(\beta_i) \), and since \( w_i = u_i(\beta_i) \) we have

\[
w_i - u_j(\beta_i) = P_i(\beta_i) - \gamma_i c(\beta_i) - (P_j(\beta_i) - \gamma_j c(\beta_j)) = (\gamma_j - \gamma_i) c(\beta_i)
\]

Similarly, \( w_i - w_j = (\gamma_j - \gamma_i) c(T) \). Since \( u_j(\beta_i) \leq w_j \) and \( c(\beta_i) < c(T) \), we have

\[
(\gamma_j - \gamma_i) (c(T) - c(\beta_i)) \leq 0
\]

which implies that \( \gamma_j \leq \gamma_i \), or \( w_i \leq w_j \). Since \( G_m(0) > 0 \) (Lemma 3), \( \beta_{m+1} = 0 \).

C.8 Proof of Proposition 6

Since cost functions equal 0 at 0 and are weakly increasing, for \( G_1 \) and \( G_2 \) to be cumulative distribution functions it suffices to show that \( G_1(T) = G_2(T) = 1 \). This equality follows from the definitions of \( T \) and \( w_1 \). It is immediate to see that \( (G_1, G_2) \) is an equilibrium.

By Lemma 9 and the Generalized Threshold Lemma, when costs are strictly increasing both players must be indifferent among all scores in \( (0, T) \), so an equilibrium has the form \( \left( \frac{w_2 + c_2(\cdot)}{v_2}, \frac{w_1 + c_1(\cdot)}{v_1} \right) \) on \( [0, T] \). Since \( T = r_2 \) and the Generalized Threshold Lemma holds, players' payoffs are \( w_1 \) and 0, respectively, pinning down the equilibrium.
C.9 The Example of Section 6

Cost functions are $c_2(x) = \frac{3x}{4}$.

$$c_1(x) = \begin{cases} \frac{x}{100} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{100} + 1.0581(x - (0.31948))^2 & \text{if } 0.31948 < x \leq 1 \\ 0.5 + 1.45(x - 1) & \text{if } 1 < x \end{cases}$$

and

$$c_3(x) = \begin{cases} \frac{x}{12} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{12} + 1.9794(x - (0.31948))^2 & \text{if } 0.31948 < x \leq 0.7259 \\ 0.38744 + 1.6923(x - 0.7259) + 25(x - 0.7259)^2 & \text{if } 0.7259 < x \leq 0.85 \\ 0.98247 + \frac{(1 - 0.98247)}{0.15}(x - 0.85) & \text{if } 0.85 < x \end{cases}$$

These cost functions give powers of $0, \frac{1}{4}$, and $\frac{1}{2}$. Perturbing the cost functions slightly does not change the qualitative aspects of the equilibrium.