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in a Calibrated One-Sector Growth Model**

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# Income Effects and Indeterminacy in a Calibrated One-Sector Growth Model

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## Abstract

This paper analyzes how the indeterminacy of competitive equilibrium in one-sector growth models depends on the magnitude of the households' income effect on the demand for leisure. The paper first establishes that the presence of income effect is necessary for the existence of an indeterminate equilibrium. Because I am further interested in quantitatively characterizing regions of uniqueness and regions of indeterminacy of equilibria as a function of this income effect, I need a utility function that is capable of inducing varying degrees of such effects. The most widely used utility functions in the business cycle literature – King, Plosser, and Rebelo (1988) (KPR) and Greenwood, Hercowitz, and Huffman (1988) (GHH) – are not suitable for this task, because they induce two polar cases of constant income effect. Therefore, I incorporate into the analysis the Jaimovich and Rebelo (2006) preferences that nest the KPR and GHH utility functions and span the entire range of income effect that exists between the two. Having identified these regions of indeterminacy, I find a lower and an upper bound for the magnitude of income effect that leads to indeterminacy. Moreover, by allowing for variation in the degree of income effect, I find that indeterminacy can occur for levels of aggregate-returns-to-scale that are well within recent empirical estimates. Finally, for these regions of indeterminacy, I simulate the model driven solely by sunspot shocks. I find that the second-moment properties of this model are generally consistent with the U.S. data at the business cycle frequency.

**Keywords:** Indeterminate Equilibria, Utility Function, Sunspot Shocks, Business Cycles.

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# 1 Introduction

The goal of this paper is to investigate how the existence of equilibrium indeterminacy depends on the magnitude of income effect on the demand for leisure.<sup>1</sup> Specifically, I study a class of one-sector infinite-horizon models of capital accumulation and external effects in production with bounded returns in which the representative agent values consumption and leisure.<sup>2</sup> Within the literature that has addressed the issue of indeterminacy in this class of models, Boldrin and Rustichini (1994) and Benhabib and Farmer (1994) are the two most relevant references.<sup>3</sup> The first paper analyzes a model in which the representative agent inelastically supplies a fixed amount of labor and only values consumption. The authors show that the equilibria that converge to the stationary state are locally unique. In the second paper, Benhabib and Farmer (1994) analyze a Cobb-Douglas economy with endogenous labor supply; they prove the existence of equilibrium indeterminacy.<sup>4</sup> The specific utility function that they analyze is characterized by the presence of a positive income effect on the demand for leisure.

Early criticism of the Benhabib and Farmer (1994) model questioned the empirical plausibility of its intermediacy result because it required a level of aggregate returns to scale (ARTS) in the production function that was at odds with the existing estimates. Subsequent work within this area<sup>5</sup> has resulted in examples of model economies that are characterized by indeterminacy with much lower and more empirically plausible levels of ARTS.<sup>6</sup>

While these subsequent studies analyzed different economic environments, one common feature they shared was the assumption of a utility function characterized by the presence of positive income effect on the demand for leisure. In this paper, I shift the focus of attention from the dependency

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<sup>1</sup>Like Boldrin and Rustichini (1994), I refer to an equilibrium indeterminacy as an equilibrium in which multiple paths converge toward the same steady state.

<sup>2</sup>All the results in the paper refer to this class of models.

<sup>3</sup>In addition to these two references, Kehoe, Levine, and Romer (1991), Kehoe (1991), and Spear (1991) among others have grappled with the issue of indeterminacy in the one-sector growth model with bounded returns. The differences among these papers lie in the type of the external effect considered.

<sup>4</sup>The Cobb-Douglas economy that Benhabib and Farmer (1994) consider is encompassed in Boldrin and Rustichini's (1994) analysis.

<sup>5</sup>See, for example, Benhabib and Farmer (1996), Benhabib and Nishimura (1998), Wen (1998), Bennet and Farmer (2000), Harrison (2001), and Harrison and Weder (2002).

<sup>6</sup>See Shell (1987) and Benhabib and Farmer (1999) for two excellent surveys of models with sunspot equilibria.

of indeterminacy on the degree of ARTS to the dependency of indeterminacy on the magnitude of income effect on the demand for leisure. Specifically, I analyze this interaction in a model otherwise identical to the one analyzed in Benhabib and Farmer (1994).

The first result that emerges from this analysis is as follows: when the utility function belongs to the class of functions that exhibit no income effect on the demand for leisure, an indeterminate equilibrium cannot exist. That is, in models with endogenous labor supply, the presence of income effect on the demand for leisure is a necessary condition for the existence of indeterminacy. Given the prevalence of utility functions that exhibit no income effect in the macroeconomic literature, this result is of particular interest.<sup>7,8</sup>

However, the quantitative implications of this necessary condition are somewhat limited. For example, it is not informative in terms of the minimum degree of income effect required for indeterminacy to exist. Similarly, this result does not allow for characterization of regions of uniqueness and regions of indeterminacy of equilibria as a function of varying degrees of income effect. Moreover, a trade-off is to be expected: an income effect that is "too strong" would tend to reduce the plausibility of equilibrium indeterminacy as this would, all other things equal, reduce the labor supply of agents and impinge on the plausibility of indeterminacy. From this quantitative point of view, it is evident that these shortcomings are driven by the fact that the two classes of utility functions most widely used in the macro literature – the class of preferences discussed in King, Plosser, and Rebelo (1988) (KPR) and the Greenwood, Hercowitz, and Huffman (1988) (GHH) utility function – represent two polar cases of constant income effect. As such, they do not enable us to conduct this type of analysis. Therefore, in order to characterize the dependence of inde-

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<sup>7</sup>A survey of the extensive use of this utility function is beyond the scope of this paper. In general, the literature has emphasized that the "no income effect" utility functions improve the ability of various models to reproduce some business cycle facts. For some prominent examples, see Greenwood, Hercowitz, and Huffman (1988) who analyze business cycle dynamics using a momentary utility function exhibiting no income effect. Mendoza (1991) introduces a generalized version of the utility function used in Greenwood, Hercowitz, and Huffman (1988) in order to examine business cycle dynamics in a small open economy. Correia, Neves, and Rebelo (1995) analyze the business cycle dynamics in a small open economy with respect to two different utility functions: one similar to that of Greenwood, Hercowitz, and Huffman (1988) and one belonging to the class described by King, Plosser, and Rebelo (1988). Perri and Neumeyer (2005) use the utility function in Greenwood, Hercowitz, and Huffman (1988) to analyze the importance of movements in the interest rate in explaining business cycle dynamics in small open economies.

<sup>8</sup>From here on, I will refer to this simply as "income effect."

terminacy on different degrees of income effect, it is necessary first to introduce a utility function flexible enough to encompass varying degrees of income effect.

The preferences introduced in Jaimovich and Rebelo (2006) (JR) turn out to be suitable candidates for this task. The JR preferences nest as special cases the KPR and GHH utility functions. Moreover, they are flexible enough that they can easily span the entire range of income effect that lies between these two utility functions. Introducing the JR utility function into the analysis allows me to quantitatively characterize regions of indeterminacy as a function of varying degrees of income effect and ARTS. Based on this characterization, I find a lower and an upper bound for the magnitude of income effect consistent with indeterminacy. Further, I show that there is a trade-off in the effect of income effect on the existence of equilibrium indeterminacy. By allowing for varying degrees of income effect, I find that indeterminacy occurs for levels of ARTS that are well within the recent empirical estimates. Finally, for these regions of indeterminacy, I simulate the model driven solely by sunspot shocks. I find that the second-moment properties of this model are generally consistent with the U.S. data at the business cycle frequency. These results suggest that the one-sector growth model can exhibit indeterminacy and generate second moment that are consistent with those observed in the U.S. data for plausible parameter values once varying degrees of income effect are introduced into the analysis.

The rest of the paper is organized as follows. Section 2 begins with a simple example showing how an indeterminate equilibrium cannot arise when the momentary utility function of the representative agent is characterized by the absence of income effect on the demand for leisure – specifically, I study the GHH utility function (I generalize the results from Section 2 in Appendix A). Section 3 introduces the Jaimovich-Rebelo preferences and shows how they are able to encompass varying degrees of income effect. I then continue by analyzing the interaction between these varying degrees of income effect and the existence of indeterminacy. The section concludes by evaluating the second moment properties of the model for different degrees of income effect and ARTS. Section 4 concludes. Appendix B analyzes the case of negative income effect on the demand for leisure. All the proofs are presented in Appendix C.

## 2 An Example

In this Section, I analyze a simple example that illustrates how the presence of income effect on the demand for leisure is a necessary condition for the existence of indeterminacy. The one-sector infinite horizon model of capital accumulation and bounded returns, in which the representative agent values consumption and leisure, analyzed in Benhabib and Farmer (1994) is characterized by the existence of indeterminacy. In order to isolate the role of income effect on the demand for leisure, I use the same model as Benhabib and Farmer (1994), with one difference: the utility function of the representative agent is characterized by no income effect on the demand for leisure. Specifically, I use the utility function introduced by Greenwood, Hercowitz, and Huffman (1988).

### 2.1 Preferences

At each point in time, the representative agent maximizes his utility from streams of consumption and leisure according to

$$\max_{H_t, C_t} \int_0^{\infty} \log \left( C_t - \frac{H_t^{1+\chi}}{1+\chi} \right) e^{-\rho t} dt \quad (2.1)$$

subject to the law of motion of capital

$$\dot{K}_t = (r_t - \delta)K_t + W_t H_t - C_t + \Pi_t \quad (2.2)$$

where  $C_t$ ,  $K_t$ , and  $H_t$  denote consumption, capital holdings, and hours worked, respectively, at period  $t$ . The time endowment is normalized to one,  $\rho$  denotes the discount rate, and  $\delta$  is the depreciation rate. The households own the capital stock and take the equilibrium rental rate,  $r_t$ , and the equilibrium wage,  $W_t$ , as given. The parameter  $\chi$  governs the labor supply elasticity with respect to the wage rate. Finally, the households own the firms and receive any profits,  $\Pi_t$ , generated.<sup>9</sup>

Denoting the Lagrangian multiplier by  $\Lambda_t$ , the set of first-order conditions characterizing the

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<sup>9</sup>The momentary utility function in (2.1) is not consistent with a balanced growth path. The common practice in the literature is to assume that the dis-utility of work in the market must increase with the level of technical progress. That is, the utility function is modified as  $U(C_t, H_t) = \log \left( C_t - Z_t \frac{H_t^{1+\chi}}{1+\chi} \right)$  where  $Z_t$  is the deterministic level of technology. The economy then would be transformed into a stationary one by rescaling the variables in the model. Following this approach will not change any of the results in the paper.

agent's optimization problem is

$$\frac{1}{C_t - \frac{H_t^{1+\chi}}{1+\chi}} = \Lambda_t \quad (2.3)$$

$$\frac{H_t^\chi}{C_t - \frac{H_t^{1+\chi}}{1+\chi}} = \Lambda_t W_t \quad (2.4)$$

$$\dot{\Lambda}_t = \Lambda_t(\delta + \rho - r_t) \quad (2.5)$$

Substituting (2.3) into (2.4) gives rise to the following IntraEuler equation

$$H_t^\chi = W_t \quad (2.6)$$

Note that the supply of hours worked is not a function of the marginal utility of wealth,  $\Lambda_t$ , but rather is an increasing function of the wage rate in the economy. Thus, the momentary utility function  $\log\left(C_t - \frac{H_t^{1+\chi}}{1+\chi}\right)$  is characterized by the presence of no income effect on the demand for leisure (see Greenwood, Hercowitz, and Huffman (1988)). Equations (2.2) and (2.5), together with the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\rho t} \Lambda_t K_t = 0 \quad (2.7)$$

and the initial condition,  $K_0$ , describe the equilibrium dynamics of the model economy.

## 2.2 Technology

The assumptions with respect to the firm's production function are identical to the ones in Benhabib and Farmer (1994). Specifically, the firm's technology is given by

$$x_t = K_t^q H_t^b \left[ \bar{K}_t^{-q\theta_1} \bar{H}_t^{b\theta_2} \right] \quad (2.8)$$

where  $K_t$  and  $H_t$  denote capital and hours worked, employed by the firm at period  $t$ .  $\bar{K}_t$  and  $\bar{H}_t$  denote aggregate capital and aggregate hours worked, respectively, at period  $t$ . The economy is assumed to consist of a large number of identical firms, and, from the perspective of the single firm,  $\bar{K}_t$  and  $\bar{H}_t$  are exogenous and represent external effects which are not traded in the markets. From the firm's viewpoint, the technology exhibits constant returns to scale, i.e.,

$$q + b = 1 \quad (2.9)$$

In a symmetric equilibrium,  $K_t = \bar{K}_t$  and  $H_t = \bar{H}_t$ , implying that the aggregate output is given by,

$$Y_t = K_t^\alpha H_t^\beta \quad (2.10)$$

where

$$\alpha = q(1 + \theta_1) > q \quad (2.11)$$

$$\beta = b(1 + \theta_2) > b \quad (2.12)$$

Since I am interested in the case of bounded returns, I assume that  $\alpha < 1$ .

**Demand for the Factors of Production** From the firm's maximization problem, it follows that

$$W_t = b \frac{Y_t}{H_t} \quad (2.13)$$

$$r_t = q \frac{Y_t}{K_t} \quad (2.14)$$

where  $W_t$  is the wage and  $r_t$  is the rental rate of capital. It therefore follows from (2.6) and (2.13) that

$$H_t^x = b \frac{Y_t}{H_t} \quad (2.15)$$

### 2.3 Dynamic System

I consider  $\lambda_t = \log \Lambda_t$  and  $k_t = \log K_t$ . Let a starred variable denote its value in the steady state and the circumflex above a variable denote the percentage deviation from the steady state. Appendix C shows that the resulting Jacobian of the dynamic system is

$$\begin{array}{|c|} \hline \dot{k}_t \\ \hline \dot{\lambda}_t \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline J_{11} & J_{12} & \widehat{k}_t \\ \hline J_{21} & J_{22} & \widehat{\lambda}_t \\ \hline \end{array}$$

where

$$J_{11} = \left( \frac{\delta + \rho}{q} \right) \left( \frac{\beta - (1 - \alpha)(1 + \chi)}{1 + \chi - \beta} \right) + \left( \frac{\delta(1 - q) + \rho}{q} \right) \left( 1 + \frac{\alpha H^{*1+x}}{C^*(1 + \chi - \beta)} \right)$$

$$J_{12} = \left( \frac{\delta(1 - q) + \rho}{q} \right) \left( 1 - \frac{H^{*1+x}}{C^*(1 + \chi)} \right)$$

$$J_{21} = -(\delta + \rho) \left( \frac{\beta - (1 - \alpha)(1 + \chi)}{1 + \chi - \beta} \right)$$

$$J_{22} = 0$$



The steady state is indeterminate if and only if the determinant is positive and the trace negative.<sup>10</sup> By analyzing the determinant, it can be shown that its sign is defined by the sign of

$$\frac{\beta - (1 - \alpha)(1 - \chi)}{(1 + \chi - \beta)} \quad (2.16)$$

which implies that the determinant is positive if and only if

$$\beta \in ((1 - \alpha)(1 + \chi), (1 + \chi)) \quad (2.17)$$

The trace is negative if and only if

$$J_{11} < 0$$

Given the admissible values for  $\beta$  that (2.17) identifies, it follows that the expression

$$\left( \frac{\delta(1 - q) + \rho}{q} \right) \left( 1 + \frac{\alpha H^{1+\chi}}{C(1 + \chi - \beta)} \right)$$

is positive. Thus, the trace is negative only if the following condition holds

$$\left( \frac{\delta + \rho}{q} \right) \left( \frac{\beta - (1 - \alpha)(1 - \chi)}{(1 + \chi - \beta)} \right) < 0$$

Given the admissible values for  $\beta$  that (2.17) identifies, this last expression is positive. Thus the trace cannot be negative whenever the determinant is positive, implying that indeterminacy cannot exist in this mode.

It is worth mentioning once again that the only difference with respect to the environment analyzed in Benhabib and Farmer (1994) is the assumption of a utility function that is characterized by the presence of no income effect on the demand for leisure. Thus, to summarize, this example demonstrates that within the Benhabib-Farmer model, the presence of income effect on the demand for leisure is a necessary condition for equilibrium indeterminacy.

One wonders, though, how robust this result is to the specific functional forms used in this example. In the Appendix of the paper, I analyze a less restrictive model that allows for identification of the general interaction between the presence of income effect and indeterminacy. The same result emerges – for a general class of utility functions and production functions indeterminacy cannot occur in the presence of no income effect on the demand for leisure.

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<sup>10</sup>The model economy has a unique steady state.

### 3 Varying Degrees of Income Effect: Implications for Indeterminacy

Given the prevalence in the macroeconomic literature of utility functions that exhibit no income effect, the necessity of a non-zero income effect is of particular interest. Moreover, it hints at the potential return to being able to analyze the interaction between equilibrium indeterminacy and income effect for varying degrees of the latter, beyond the limiting case of no income effect. However, as discussed in the introduction, the main limitation for conducting such a quantitative analysis is that the two classes of utility functions most widely used in the macro literature – the class of preferences discussed in King, Plosser, and Rebelo (1988) (KPR) and the Greenwood, Hercowitz, and Huffman (1988) (GHH) utility function – represent two polar cases of constant income effect. Therefore, this type of analysis cannot be conducted. So, in order to be able to characterize the dependence of indeterminacy on different degrees of income effect, it is necessary first to introduce a utility function flexible enough to encompass varying degrees of income effect. As mentioned earlier, the preferences in Jaimovich and Rebelo (2006) nest as special cases the KPR and GHH utility functions, and they span the entire range of income effect that lie between these two.

Specifically, the agents in the JR economy maximize a life-time utility  $U$ , which is defined over sequences of consumption and hours worked according to the following functional form<sup>11</sup>

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \psi X_t H_t^{1+\chi}\right)^{1-\sigma} - 1}{1-\sigma} \quad (3.1)$$

where

$$X_t = C_t^\gamma X_{t-1}^{1-\gamma} \quad (3.2)$$

The parameter  $\psi > 0$  is used for accounting for the steady state of  $H_t$ .  $\sigma > 0$  controls the curvature of the utility function. For the rest of the paper I concentrate at the case of  $\sigma = 1$ . When  $\gamma = 1$ , these preferences belong to the KPR class of preferences.

When  $\gamma = 0$ , the GHH preferences are obtained. Also, the JR preferences are consistent with balanced growth in the presence of labor-augmenting or investment-specific technical progress, and they exhibit weak income effect on leisure in the short run for low values of  $\gamma$ . For the purposes of

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<sup>11</sup>As I am interested in eventually simulating the model (see below) and comparing its predictions to the U.S. data, I shift to a discrete-time model.

this paper, these preferences allow me to control the magnitude of income effect by varying  $\gamma$  for values between the two extremes (i.e.,  $\gamma \in [0, 1]$ ). Thus I can characterize regions of indeterminacy as a function of income effect and ARTS.

Before performing this analysis, I discuss some general features of this utility function that will be useful for the analysis of the interaction between income effect and indeterminacy.

### 3.1 Dynamic Hicksian Decomposition of Income Effect

In order to investigate the nature of the income effect and how it depends on the value of  $\gamma$ , it would be useful to isolate the Hicksian income effect on the labor supply. As in Jaimovich and Rebelo (2006), I follow here the approach in King (1991), which discusses a dynamic version of the Hicks decomposition. This experiment is carried out as follows. First, I study the response of hours of an agent who faces a permanent increase of 1% in  $TFP$ .<sup>12</sup> For each value of  $\gamma \in [0, 1]$ , this permanent shock raises the lifetime utility from  $U^*$  to  $U^*(\gamma)$ .<sup>13</sup> Figure 1 graphs the response of hours worked to the permanent  $TFP$  increase as a function of 10 different values of  $\gamma$  over 25 periods. The strongest response of hours worked occurs with GHH preferences ( $\gamma = 0$ ). In this case, because of the lack of income effect, hours worked are not stationary – they rise permanently in response to the permanent increase in the real wage rate. With KPR preferences ( $\gamma = 1$ ), hours worked converge back to the steady state after the shock, but the short-run response of hours worked is weak.<sup>14</sup> The other lines represent the response of hours worked when  $\gamma \in (0, 1)$ . With these preferences, hours worked also converge to the steady state, but the short-run impact of the  $TFP$  shock falls between those of GHH and KPR preferences.<sup>15</sup> Lower (higher) values of  $\gamma$  produce short-run responses that are closer to those obtained with GHH (KPR) preferences.

Then, to calculate the income effect, I compute, for each  $\gamma \in [0, 1]$ , the path of labor supply of a household that receives an output transfer and faces wages and real interest rates that are constant

<sup>12</sup>To isolate the effects of variations in  $\gamma$ , I first study this decomposition for the case of an aggregate production function that exhibits constant returns to scale. See below for a similar decomposition for different levels of ARTS.

<sup>13</sup>In order to have the steady-state allocations not depend on  $\gamma$ , and thus  $U^*$  remain the same prior to the  $TFP$  shock,  $\psi$  has to be a function of  $\gamma$ ,  $\psi(\gamma) = \frac{1}{(1+\chi)(H^*)^\chi \frac{C^*}{W^*} + \frac{\gamma(H^*)^{1+\chi}}{1-\rho(1-\gamma)}}$ .

<sup>14</sup>For  $\gamma = 1$ , the convergence occurs after 50 periods.

<sup>15</sup>As long as  $0 < \gamma \leq 1$ , hours worked converge back to the steady state. The slowest convergence occurs for the lowest value of  $\gamma > 0$  that is considered –  $\gamma = 0.01$ . This takes place after 150 periods.

at their steady state levels. The level of the transfer is computed such that the agent's utility is  $U^*(\gamma)$  (without the transfer the agent's utility would be  $U^*$ ). Thus, the agent solves the following Lagrangian

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \psi X_t H_t^{1+\chi}\right)^{1-\sigma} - 1}{1-\sigma} \\ & + \sum_{t=0}^{\infty} \beta^t \lambda_t (W_t H_t + (1+r_t)A_t + \zeta_t - C_t - A_{t+1}) \\ & + \sum_{t=0}^{\infty} \beta^t \phi_t \left(X_t - C_t^\gamma X_{t-1}^{1-\gamma}\right) \end{aligned}$$

where  $\zeta_t$  is the output transfer and  $A_t$  denotes wealth at period  $t$ .

Based on this dynamic Hicksian decomposition, Figure 2 graphs the response of hours that are attributable to the income effect for the same 10 values of  $\gamma$  shown in Figure 1. For the GHH preferences, the income effect is zero, whereas for the KPR preferences the income effect is negative (in both cases the income effect is constant over time). With the JR preferences, the income effect is time varying. Like the total response of hours, for  $0 < \gamma \leq 1$ , the income effect converges to that of the KPR preferences in the long run.<sup>16</sup>

Figure 2 reveals two interesting results. First, in the short-run, there is a hump-shape in the income effect as a function of  $\gamma$ . Second, for certain values of  $\gamma$ , the short-run income effect is actually negative (i.e., inducing a short-run increase in hours worked). These two effects are easier to notice in Figure 3 which graphs, for each different level of  $\gamma$ , the income effect at the period of the transfer shock. Again, the hump-shaped income effect is evident from this figure. The rationale for the negative income effect is the following. Consumption grows over time in this experiment. With this type of preferences, the growth in consumption implies that the dis-utility of work is higher in the future than in the present – for relatively lower values of  $\gamma$ , the future consumption growth is contemporaneously weighted more heavily, inducing an increase in hours worked.

Now I turn to the implications of variation in  $\gamma$ , and thus to variations in the strength of the income effect, for the existence of indeterminacy.

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<sup>16</sup>The asymptotic convergence for the case of  $\gamma = 0.01$  occurs after more than 50 periods.

### 3.2 Discussion of the Implications for Indeterminacy

In order to isolate the role of income effect on the existence of indeterminacy, I use the same model as Benhabib and Farmer (1994), with one difference. The utility function of the representative agent is given by the JR preferences, as specified in (3.1) and (3.2). The rest of the model is identical to the one analyzed in Section 2, which is the one in Benhabib and Farmer (1994). I will refer to this model as the Benhabib-Farmer-Jaimovich-Rebelo (BFJR) model.

I begin the analysis by characterizing regions of indeterminacy for various levels of ARTS,  $(\alpha + \beta)$ , and different values of  $\gamma$ . Since a point estimator of  $\gamma$  does not exist at the literature<sup>17</sup> I perform this analysis for various levels of  $\gamma \in [0, 1]$ . It is worth emphasizing again, that any value of  $\gamma > 0$  is consistent with the stationarity of hours in the U.S. data and with balanced growth in the presence of labor-augmenting or investment-specific technical progress; Thus, these empirical considerations by themselves cannot rule out any value of  $\gamma$ .<sup>18</sup> This should be thus regarded as a first step in studying the quantitative importance of income effects for indeterminacy.

The dark region in Figure 4 emphasizes the main message of this paper: the analysis of the interaction between income effect and indeterminacy yields new insights on the plausibility of the latter.<sup>19</sup> First of all, Figure 4 shows that, indeed, for any level of ARTS there is no indeterminate equilibrium for the case of  $\gamma = 0$  (i.e., the case of no income effect).<sup>20</sup> Moreover, Figure 4 reveals that there is a minimum value of  $\gamma$  that is required such that an indeterminate equilibrium exists.<sup>21</sup> The intuition for this result is similar in nature to the discussion in the introduction. In order for indeterminacy to exist, there has to be a strong enough reaction of the labor supply to the sunspot shock, because the labor demand curve is fixed at the period of the shock. However, as discussed in the introduction, a trade-off is to be expected: when the "aggregate labor demand" is either downward sloping (region *A* of Figure 4), or upward sloping but with a smaller slope than that of the labor supply (region *B* of Figure 4), then an income effect that is "too strong" would tend to

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<sup>17</sup>In on going work in Amador, Jaimovich, and Rosen (2007) we estimate  $\gamma$  using a variety of methods and data sets.

<sup>18</sup>See in the next section the discussion on the stationarity of hours in the model with a finite data length.

<sup>19</sup>In the analysis that follows, I concentrate on the case of an increase in hours worked in response to a sunspot shock.

<sup>20</sup>This is true even for higher levels of ARTS than shown in Figure 4.

<sup>21</sup>In order to find the minimum value, I discretize the values of  $\gamma$  around 0 with increments of 0.001. I find that this is obtained at  $\gamma = 0.04$ .

work against the possibility of indeterminacy. Indeed, Figure 4 reveals that this is the case: for all the levels of ARTS considered in Figure 4, there is a maximum level of  $\gamma$  which is consistent with indeterminacy.<sup>22</sup> Finally, as expected, Figure 4 reveals that the higher the ARTS, the broader the range of  $\gamma$ 's for which indeterminacy exists.

Another way to glean an insight into the interaction between the income effect and indeterminacy is through analyzing the labor market equilibrium condition.<sup>23</sup> The first-order condition with respect to hours is

$$\left(C_t - \psi H_t^{1+\chi} X_t\right)^{-\sigma} \psi(1+\chi) H_t^\chi X_t = \lambda_t W_t$$

and the log-linearized first-order condition for the labor supply is

$$\hat{h}_t = \frac{1 - \psi H^{*1+\chi}}{\psi H^{*1+\chi} + \chi} \hat{w}_t + \left( \frac{(1-\gamma) \hat{c}_t + (1 - \psi H^{*1+\chi}) \hat{\lambda}_t}{\psi H^{*1+\chi} + \chi} \right) - \left( \frac{(1-\gamma)}{\psi H^{*1+\chi} + \chi} \right) \hat{x}_{t-1}$$

The coefficient of  $\hat{w}_t$  acts as the slope in this log-linearized labor supply condition, while the expression

$$\left( \frac{[1-\gamma] \hat{c}_t + [1 - \psi H^{*1+\chi\theta}] \hat{\lambda}_t}{\psi H^{*1+\chi} + \chi} \right) - \left( \frac{(1-\gamma)}{\psi H^{*1+\chi} + \chi} \right) \hat{x}_{t-1}$$

acts as a shift variable.

Assume that at period  $t$  the economy is at a steady state; thus,  $\hat{x}_{t-1} = 0$ . At the arrival of the sunspot shock (similar to Benhabib and Farmer (1994), I model the sunspot shock as a fall in  $\hat{\lambda}$ ), the response of hours is determined by the sign of

$$\left( \frac{[1-\gamma] \hat{c}_t + [1 - \psi H^\theta] \hat{\lambda}_t}{\psi H^\theta + (\theta - 1)} \right) \quad (3.3)$$

<sup>22</sup>Figure 3 shows that for values of  $\gamma > 0.04$ , increases in  $\gamma$  induce a stronger income-effect on hours. Figure 4 reveals that for all the levels of ARTS considered, the maximum level of  $\gamma$  is always in this region of  $\gamma$ 's.

<sup>23</sup>Benhabib and Farmer (1994) discuss the necessary conditions for indeterminacy in their model through the labor market equilibrium as well.

<sup>24</sup>Because of the dynamic nature of the problem,  $\lambda_t$  cannot be expressed solely as a function of period  $t$  variables. Rather, it is a function of the entire future sequence of variables. This can be seen from the combination of the first order conditions for  $C_t$  and  $X_t$

$$\phi_t - \left[ \lambda_t + \phi_t \gamma \left( \frac{C_t}{X_{t-1}} \right)^{\gamma-1} \right] \psi H_t^{1+\chi} = \beta(1-\gamma) \phi_{t+1} \left( \frac{C_{t+1}}{X_t} \right)^\gamma$$

The presence of  $\phi_{t+1}$  in this equation induces the dynamic nature of the problem.

In the numerical simulations, I consider only those cases in which the labor supply is upward sloping and the "aggregate labor demand" is either downward sloping or upward sloping but with a smaller slope than that of the labor supply. Therefore, hours can become positive at the impact period only if the labor supply shifts to the "right and downward." That is, only if the sign of (3.3) is positive at the impact period of the sunspot shock can indeterminacy occur.<sup>25</sup> It is thus of interest to consider how variations in  $\gamma$  affect the value of (3.3). Figure 5 shows how the coefficient of  $\hat{\lambda}_t$  and  $\hat{c}_t$  depend on the value of  $\gamma$ .<sup>26</sup> First, as  $\gamma$  increases, the value of the coefficient on  $\hat{\lambda}_t$  increases. Since the sunspot shock is modeled as a fall in the marginal utility of wealth, all other things being equal, an increase in  $\gamma$  reduces the plausibility of the existence of indeterminacy. Similarly, for values of  $\gamma$  above 0.06, the coefficient on  $\hat{c}_t$  decreases as  $\gamma$  increases. This again implies that, all other things being equal, an increase in  $\gamma$ , and thus a strengthening of the income effect, reduces the plausibility of indeterminacy.

### 3.2.1 Indeterminacy when "Aggregate Labor Demand" is Downward Sloping

Figure 4 reveals that allowing for a weaker income effect than the one induced by the KPR utility function implies that indeterminacy exists even in the case of ARTS that are small enough that the "aggregate labor demand" has the "conventional" slope, i.e., negative. For example, I find that the minimum level of ARTS for which there is an indeterminate equilibrium is 1.09. This value is well within the empirical estimates found in the literature.<sup>27</sup> In order to better understand the occurrence of indeterminacy in the case of "downward" labor demand, I conduct a dynamic Hicksian decomposition of the hours' response to a sunspot shock in the BFJR model. That is, for each pair of ARTS and  $\gamma$  that gives rise to an indeterminate equilibrium in area *A* in Figure 4, I introduce a sunspot shock at the first period.<sup>28</sup> The sunspot shock raises the life-time utility from

<sup>25</sup>This somewhat "informal" argument is verified numerically.

<sup>26</sup>The two coefficients are independent of the degree of ARTS, which enables me to concentrate on the effects of varying  $\gamma$ . As discussed previously, in order to ensure that the non-stochastic steady state is independent of  $\gamma$ ,  $\psi$  has to be a function of it. This is why the coefficient of  $\hat{c}$  is not equal only to the expression  $(1 - \gamma)$  and why the coefficient of  $\hat{\lambda}_t$  is a function of  $\gamma$ . Obviously, this analysis is not complete, as it abstracts from the response of consumption. The idea is to study how, holding everything else constant, variations in  $\gamma$  affect the plausibility of indeterminacy.

<sup>27</sup>See, for example, ? and Basu and Fernald (1997).

<sup>28</sup>I analyze the response to a sunspot shock and not a *TFP* shock, as done previously, for two reasons. First, I am interested in learning the response of the system to a sunspot shocks because I later study the second moment

$U^*(ARTS)$  to  $U^*(ARTS, \gamma)$ .<sup>29</sup> Then, for each of these pairs I follow the steps discussed in Section 3.1 and calculate the level of the output transfer such that the agent's utility equals  $U^*(ARTS, \gamma)$ . Based on this decomposition, Figure 6 shows the response of hours, at the moment of the impact of the sunspot shocks, which is attributable to the income effect. I graph this response for six different levels of ARTS that induce "aggregate labor demand" that is downward sloping.

Three interesting results emerge from this exercise. First, when indeterminacy occurs in the case of downward "aggregate labor demand," the response of hours at the impact period that is due to the income effect is always positive. This fact highlights why, for the KPR case ( $\gamma = 1$ ), a much higher level of ARTS is required (essentially an upward "aggregate labor demand"); for this case, the income effect at the impact period for this value of  $\gamma$  is negative (see Figure 2). Second, conditional on the level of ARTS (along a graph), when indeterminacy occurs, the higher  $\gamma$  is, the stronger the income effect (i.e., the lower is the positive response). This again reflects how increases in  $\gamma$  strengthen the income effect and, holding everything else constant, reduce the plausibility of indeterminacy. Third, conditional on the level of  $\gamma$ , the higher the level of ARTS, the weaker the income effect (i.e., the higher the value of the response of hours worked). This last effect again illustrates the previous discussion of Figure 4: the higher the ARTS, the broader the range of  $\gamma$ 's for which indeterminacy exists.

### 3.3 Second Moments

In this section I analyze the business cycle properties of the BFJR model. Specifically, I consider ten different cases of ARTS that induce "aggregate labor demand" that is downward sloping: for each of these cases I find the set of  $\gamma$ 's that gives rise to an indeterminate equilibrium. Each of these ten sets is then discretized by increments of 0.01, resulting in 136 pairs of ARTS and  $\gamma$  for which the model is indeterminate. For each of these pairs, I simulate the model and compute the second moment of output, consumption, investment, and hours.<sup>30</sup>

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properties of a model driven by this shock. Second, because the marginal utility of wealth is a jump variable, its response is not pinned down after a *TFP* shock in the case of an indeterminate equilibrium. By modeling the sunspot shock as in Benhabib and Farmer (1994), the "jump" of the marginal utility of wealth at the impact period is well defined.

<sup>29</sup>In the steady state, the allocations depend on the degree of ARTS.

<sup>30</sup>In all the simulations I follow the approach in Benhabib and Farmer (1994), and I set the volatility of the sunspot shock so that the model matches the volatility of output in the U.S. data.



Before presenting the results, it is worth addressing the stationarity of hours. As discussed previously, the unique case where the JR preferences are not consistent with balanced growth path is the case of  $\gamma = 0$ . For any other value of  $\gamma > 0$  the JR preferences are such that hours are stationary. However, one concern is that in small samples the model will generate non-stationarity of hours. I thus check for non stationarity of hours in each of the 136 simulations and find that in all the simulations the behavior of hours is consistent with balanced growth path.<sup>31</sup>

For ease in presenting the simulation results for these 136 pairs, Figure 7 graphs the standard deviation of HP-filtered hours relative to the standard deviation of HP-filtered output for all pairs.<sup>32</sup> Similarly, Figures 8 and 9 graph the standard deviation of HP-filtered consumption and HP-filtered investment relative to the standard deviation of HP-filtered output, respectively. For comparison, in the U.S. data the standard deviations of HP-filtered hours, HP-filtered investment, and HP-filtered consumption relative to the standard deviation of HP-filtered output are 0.97, 0.75, and 3.05, respectively.<sup>33</sup>

With respect to the volatility of hours, the model's performance is generally consistent with the data, as hours are as volatile as output.<sup>34</sup> Since the JR preferences are not separable across consumption and labor effort, the optimality conditions of the household problem imply that  $(C_t - \psi H_t^{1+\chi} X_t)$  should be smoothed over time. For this reason, the response of labor supply to the sunspot shock induces additional movements in consumption. Indeed, as Figure 8 suggests, the model generates volatility of consumption that is closer to the one observed in the data than that generated by other one-sector "sunspot models," all of which assume a utility function that belongs to the KPR class (see, for example, Farmer and Guo (1994) and Wen (1998). In the former the ratio equals 0.23, whereas in the latter it equals 0.04). The increased volatility of consumption in turn induces a reduction in the relative volatility of investment as compared to the results of these other models (again, for example, this equals 8.91 in Farmer and Guo (1994) and 4.63 in Wen

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<sup>31</sup>An obvious related empirical question is what is the value of  $\gamma$ ? In on going work in Amador, Jaimovich, and Rosen (2007) we estimate  $\gamma$  using a variety of methods.

<sup>32</sup>A smoothing parameter of 1600 is used.

<sup>33</sup>I use quarterly data between 1947 : I – 2004 : IV.

<sup>34</sup>It is interesting to note that, conditional on a value of  $\gamma$ , a higher level of ARTS will lead to a lower volatility of hours. This might be surprising, given the dynamic income effect decomposition carried out previously. However, a decomposition of the dynamic substitution effect can be conducted that shows that, conditional on a value of  $\gamma$ , the higher the ARTS, the lower the substitution effect, thus inducing an overall reduction in the volatility.

(1998)) and brings it closer to the observed ratio in the U.S. data.<sup>35</sup>

With respect to the contemporaneous correlations of HP-filtered hours with HP-filtered output, as in the U.S. data, the model generates high correlations: all 136 correlations fall between 0.97 and 0.99 (the correlation in the U.S. data equals 0.86). Similarly, the model generates a consumption process that is highly correlated with output: the 136 correlations lie between 0.8 and 0.99 (the correlation in the U.S. data equals 0.77). Finally, with respect to investment, the model generates an investment process that is highly correlated with output as well: the correlations lie between 0.91 and 0.97 (the correlation in the U.S. data equals 0.89).

Thus, to conclude this section, with the sole modification of the JR preferences, the model induces (1) a variance of hours, consumption, and investment, and a contemporaneous correlation between these three variables and output that in general is consistent with the U.S. data, and (2) an empirically relevant increase (decrease) in the volatility of consumption (investment) relative to one-sector "sunspot models" that assume a utility function of the KPR class. Once again, it is worth noting that these results are obtained for values of ARTS that are close to those estimated in the U.S. data and that induce "aggregate labor demand" that is downward sloping. Moreover, these results are obtained for values of  $\gamma$  that induce stationary hours and are thus consistent with the behavior of hours in the U.S. data. Thus, these results suggest that the one-sector growth model can exhibit indeterminacy and generate second moment that are consistent with those observed in the U.S. data for plausible parameter values.

## 4 Conclusions

This paper studies how the determinacy of competitive equilibrium depends on the magnitude of income effect on the demand for leisure. Specifically, I study a class of one-sector infinite horizon models of capital accumulation, and external effects in production, with bounded returns, in which the representative agent values consumption and leisure. The paper shows that, for very general utility functions and production functions, the equilibria converging to the steady state in this class of models are unique whenever the utility function is characterized by the absence of income effect.

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<sup>35</sup>Wen (1998) provides an excellent discussion on how the low volatility of consumption relative to output in his model (as well as in that of Farmer and Guo (1994)) leads to a very volatile investment path. Because the BFJR model belongs to the class of one-sector growth models that he considers, his discussion applies here as well.

The paper then quantitatively characterizes regions of uniqueness and regions of indeterminacy of equilibria as functions of the magnitude of the income effect. This demonstrates the dependence of equilibrium indeterminacy on the degree of income effect. The analysis is carried out using the Jaimovich and Rebelo (2006) utility function, which spans the entire range of income effect that exist between the two most widely used utility functions in the business cycle literature: the King, Plosser, and Rebelo (1988) (KPR) and the Greenwood, Hercowitz, and Huffman (1988) (GHH) utility functions.

The paper suggests that, when the model is calibrated with levels of aggregate returns to scale that are within recent empirical estimates, there is a wide range of degrees of income effect that lie between the KPR and GHH values, giving rise to equilibrium indeterminacy. For these regions of indeterminacy, I simulate the model driven solely by sunspot shocks. I find that the second-moment properties of this model are generally consistent with the U.S. data at the business cycle frequency. These results suggest that the one-sector growth model can exhibit indeterminacy and generate second moment that are consistent with those observed in the U.S. data for plausible parameter values.

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## Appendix A: The General Model

### Preferences

At each point in time, the representative agent maximizes his utility from streams of consumption and leisure according to

$$\max_{C_t, H_t} \int_0^{\infty} U(C_t, H_t) e^{-\rho t} dt$$

subject to the law of motion of capital

$$\dot{K}_t = (r_t - \delta) K_t + W_t H_t - C_t + \Pi_t \quad (4.1)$$

where  $C_t$ ,  $K_t$ , and  $H_t$  denote, respectively, consumption, capital holdings, and hours worked at period  $t$ . The time endowment is normalized to one,  $\rho$  denotes the discount rate, and  $\delta$  is the depreciation rate. The households own the capital stock and take the equilibrium rental rate,  $r_t$ , and the equilibrium wage,  $W_t$ , as given. Finally, the households own the firms and receive any profits,  $\Pi_t$ .

**The Utility Function** Let the momentary utility function be defined as

$$U(C_t, H_t) = U(Q(C_t) - V(H_t)) \quad (4.2)$$

I assume that the period utility function  $U$  is a continuously differentiable, strictly increasing, and strictly concave function. The strict concavity and monotonicity imply that

$$\frac{\partial U(C, H)}{\partial C} = U'(Q(C_t) - V(H_t)) Q'(C) > 0 \quad (4.3)$$

$$Q'(C) > 0 \quad (4.4)$$

$$\frac{\partial^2 U(C, H)}{\partial C^2} = U''(Q(C_t) - V(H_t)) (Q'(C))^2 + U'(Q(C_t) - V(H_t)) Q''(C) < 0. \quad (4.5)$$

Also, in order to ensure an interior solution, I assume

$$\lim_{(Q(C_t) - V(H_t)) \rightarrow 0} U(C_t, H_t) = \infty$$

I assume that the dis-utility from hours worked is increasing and weakly convex<sup>36</sup>

$$V'(H) > 0 \quad (4.6)$$

$$V''(H) \geq 0 \quad (4.7)$$

Note that conditions (4.3) through (4.7) are sufficient to imply joint concavity of U.

**First-Order Conditions** Denoting the Lagrangian multiplier by  $\Lambda_t$ , the set of first-order conditions characterizing the agent's optimization problem is

$$U'(Q(C_t) - V(H_t)) Q'(C_t) = \Lambda_t \quad (4.8)$$

$$U'(Q(C_t) - V(H_t)) V'(H_t) = \Lambda_t W_t \quad (4.9)$$

$$\dot{\Lambda}_t = \Lambda_t (\rho + \delta - r_t) \quad (4.10)$$

By combining equations (4.8) and (4.9) the following intra-temporal first order condition equation is obtained

$$V'(H_t) = Q'(C_t) W_t \quad (4.11)$$

Equations (4.1) and (4.10), together with the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\rho t} \Lambda_t K_t = 0 \quad (4.12)$$

and the initial condition,  $k_0$ , describe the equilibrium dynamics of the model economy.

**Proposition 1** *The sign of  $Q''(C)$  determines the nature of the income effect. Specifically, depending on the sign of  $Q''(C)$ , there are three different cases to consider:*

- (i)  $Q''(C) = 0$  : No income effect.
- (ii)  $Q''(C) > 0$  : Negative income effect.
- (iii)  $Q''(C) < 0$  : Positive income effect.

## Technology

The representative firm's technology is given by

$$x_t = G(K_t, H_t, \bar{K}_t, \bar{H}_t) \quad (4.13)$$

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<sup>36</sup>The utility function defined in Section 2 satisfies the assumptions stated in equations (4.3) to (4.7).

where  $K_t$  and  $H_t$  denote the capital and hours worked, used by the firm, at period  $t$ .  $\bar{K}_t$  and  $\bar{H}_t$  denote the aggregate capital and aggregate hours worked at period  $t$ , respectively. The economy is assumed to consist of a large number of identical firms and, from the perspective of the firm,  $\bar{K}_t$  and  $\bar{H}_t$  are exogenous and represent external effects that are not traded in the markets. From the viewpoint of the firm, the technology exhibits constant returns to scale, i.e.,

$$G(\psi K_t, \psi H_t, \bar{K}_t, \bar{H}_t) = \psi G(K_t, H_t, \bar{K}_t, \bar{H}_t) \quad (4.14)$$

In equilibrium,  $K_t = \bar{K}_t$ ,  $H_t = \bar{H}_t$ . Let  $Y_t$  denote the aggregate output; thus

$$Y_t = G(\bar{K}_t, \bar{H}_t, \bar{K}_t, \bar{H}_t) \quad (4.15)$$

The firm's maximization problem implies

$$G_1(K_t, H_t, \bar{K}_t, \bar{H}_t) = r_t \quad (4.16)$$

$$G_2(K_t, H_t, \bar{K}_t, \bar{H}_t) = W_t \quad (4.17)$$

I assume that the production function  $G(\cdot)$  is characterized by the following properties: (i) the marginal productivities of capital and hours owned by the firm are positive and decreasing; (ii) the marginal productivity of capital owned by the firm is increasing with the amount of hours used by the firm, and vice versa; (iii) the firm needs its own capital and labor in order to produce<sup>37</sup>; and (iv) since I am interested in the case of positive externalities, I assume that aggregate capital and aggregate hours increase the marginal productivity of capital and hours used by the firm.

In order to ensure an interior solution, I assume the following Inada conditions

$$\begin{aligned} \lim_{H_t \rightarrow 0} G(K_t, H_t, \bar{K}_t, \bar{H}_t) &= \infty \\ \lim_{K_t \rightarrow 0} G(K_t, H_t, \bar{K}_t, \bar{H}_t) &= \infty. \end{aligned}$$

Like Boldrin and Rustichini (1994), I am interested in analyzing the case of bounded returns. The aggregate production function  $G(\bar{K}_t, \bar{H}_t, \bar{K}_t, \bar{H}_t)$  is restricted so as to impede growth when the

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<sup>37</sup>That is,

$$G(0, H_t, \bar{K}_t, \bar{H}_t) = G(K_t, 0, \bar{K}_t, \bar{H}_t) = 0.$$



following condition is satisfied

$$G_{11}(K_t, H_t \bar{K}_t, \bar{H}_t) + G_{13}(K_t, H_t \bar{K}_t, \bar{H}_t) < 0.^{38}$$

I denote  $\lambda_t = \log \Lambda_t$  and  $k_t = \log K_t$ . Appendix C shows that the resulting Jacobian of the dynamic system is given by

$$\begin{array}{|c|} \hline \dot{k}_t \\ \hline \dot{\lambda}_t \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline J_{11} & J_{12} & \widehat{k}_t \\ \hline J_{21} & J_{22} & \widehat{\lambda}_t \\ \hline \end{array} \quad (4.18)$$

where

$$\begin{aligned} J_{11} &= (A_7 - 1) \frac{Y^*}{K^*} - (A_{10} - 1) \frac{C^*}{K^*} \\ J_{12} &= A_8 \frac{Y^*}{K^*} - A_9 \frac{C^*}{K^*} \\ J_{21} &= -K^* (G_{11}^* + G_{13}^*) - H^* A_6 (G_{12}^* + G_{14}^*) \\ J_{22} &= -H^* (G_{12}^* + G_{14}^*) A_5 \end{aligned}$$

and  $A_5$  to  $A_{10}$  are functions of the parameters in the model (these functions are defined in Appendix C).

The determinant of the dynamic system is given by

$$Det = \left\{ \begin{array}{l} (G_{12} + G_{14}) \left( \begin{array}{l} \frac{Y^* H^*}{K^*} (A_5 (1 - A_7) + A_6 A_8) \\ + \frac{C^* H^*}{K^*} ((A_{10} - 1) A_5 - A_6 A_9) \end{array} \right) \\ + (G_{11} + G_{13}) (A_8 Y^* - A_9 C^*) \end{array} \right\} \quad (4.19)$$

and the trace is given by

$$Trace = (A_7 - 1) \frac{Y^*}{K^*} - (A_{10} - 1) \frac{C^*}{K^*} - H^* (G_{12} + G_{14}) A_5 \quad (4.20)$$

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<sup>38</sup>To observe this, notice that a balanced growth path exists if and only if  $G_{11} + G_{13} = 0$ . If  $G_{11} + G_{13} > 0$ , then the economy grows at variable rates. Also, the production function analyzed in Section 2 satisfies all of the assumptions about the production function given here. Specifically, with respect to the "no growth condition," it is trivially satisfied in the model analyzed in Section 2 as

$$G_{11} + G_{13} = q(q-1) \frac{Y}{K^2} + qq\theta_1 \frac{Y}{K^2} = (\alpha-1)q \frac{Y}{K^2} < 0.$$

## No Income Effect on the Demand for Leisure

As discussed in Proposition 1, the case in which the utility function is characterized by the presence of no income effect implies

$$Q''(C) = 0 \tag{4.21}$$

This leads to the following proposition.

**Proposition 2** *When the utility function belongs to the class of preferences of studies in this Appendix and it is characterized by the presence of no income effect on the demand for leisure, there cannot exist an indeterminate equilibrium.*

The economic intuition I can offer for the last proposition is as follows: indeterminacy is merely the existence of multiple equilibrium trajectories in the  $\{k_t, \lambda_t\}$  planar system. This implies that, for a given value of capital, there are infinitely many values of the marginal utility of wealth that satisfy the equilibrium conditions. The IntraEuler equation is given by

$$V'(H_t) = Q'(C)W_t$$

and, in the case of no income effect,

$$V'(H_t) = BW_t$$

where  $B$  is a constant. As is clear from this expression, in the presence of no income effect, any jumps in consumption do not affect the labor supply curve. This implies that for a given capital stock, the labor supply curve has a unique position, and hence there is a unique level of output. Given the unique level of output, the indeterminacy of equilibrium is only possible if agents alter their shares of consumption and investment out of total output. Assume, for example, that agents become optimistic (i.e., there is a "positive shock to expectations") and expect a fall in the marginal utility of wealth. Recall that the marginal utility of wealth was derived as

$$U'(Q(C_t) - V(H_t))B = \Lambda_t$$

Because  $H_t$  is uniquely determined at the shock's impact period, a fall in the marginal utility of wealth,  $\Lambda_t$ , can exist only if consumption,  $C_t$ , rises and investment falls. The decrease in investment implies that the rental rate of capital increases in the following period; from the Inter-Euler equation

it follows that the marginal utility of wealth must decrease in the following period. Notice that since investment fell, in the subsequent period capital will fall as well, and hours also must fall, reducing output and investment even further. Trajectories for investment of this sort can be maintained for a number of periods, but the resulting under-accumulation of capital eventually will violate the transversality condition because, at some time in the future, the agent will stop investing further, implying that consumption eventually will collapse, violating the Inter-temporal condition.

### Negative Income Effect on the Demand for Leisure

In their model, Benhabib and Farmer (1994) identify a necessary condition for the existence of an indeterminate equilibrium in their model. They then show that it can be expressed as follows: the magnitude of the elasticity of equilibrium wage rate with respect to aggregate hours must be higher than the elasticity of labor supply with respect to the wage rate. In the interest of completing the analysis carried out in this Appendix, I analyze the case of negative income effect on the demand for leisure and show that Benhabib and Farmer's (1994) result applies here as well, with one modification: in the case of negative income effect, the elasticity of equilibrium wage rate with respect to aggregate hours must be smaller than the elasticity of labor supply with respect to the wage rate.

**Proposition 3** *Denote by  $\eta_{V,H}^*$  the elasticity of labor supply*

$$\eta_{V,H}^* = \frac{dV'(H)}{dH} \frac{H}{V'(H)} \Big|_{H^*}$$

*and let  $\eta_{W,H}^*$  denote the elasticity of the equilibrium wage rate with respect to aggregate hours*

$$\eta_{W,H}^* = \frac{dW}{dH} \frac{H}{W} \Big|_{H^*,K^*}$$

*It then follows that  $\eta_{W,H}^* > \eta_{V,H}^*$ , if and only if  $G_{22} + G_{24} > \frac{V''}{Q'}$ , and that  $\eta_{W,H}^* < \eta_{V,H}^*$ , if and only if,  $G_{22} + G_{24} < \frac{V''}{Q'}$ .*

**Proposition 4** *With negative income effect on the demand for leisure, ( $Q''(C^*) > 0$ ), an indeterminate equilibrium exists only if  $\eta_{W,H}^* < \eta_{V,H}^*$ .*

The economic intuition I can offer for the last proposition is identical in spirit to the discussion in Section 4. At the impact period of the positive shock to expectations, an increase in consumption

is accompanied by a reduction in the desired consumption of leisure. In order to avoid the sort of trajectory discussed in Section 4, the reduction in demanded leisure must be accompanied by an actual increase in output. This process takes place only if the following condition holds: the elasticity of equilibrium wage rate with respect to aggregate hours is lower than the elasticity of labor supply. With regard to the discussion in Benhabib and Farmer (1994), this condition implies that in the presence of negative income effect on the demand for leisure, a necessary condition for indeterminacy is that the labor supply curve has to cut the "aggregate labor demand" (which is the locus of equilibrium hours and wages pairs) from below when the locus is upward sloping, or that the aggregate labor demand curve must be downward sloping. This implies that indeterminacy can exist in the case of constant ARTS.

## Appendix B: Proofs

**Proof.** [Proposition 1] Follows immediately from assumptions (4.6),(4.7), and (4.11) ■

**Proof.** [Proposition 2] Substituting (4.21) into (4.40), (4.46), and (4.50), it follows that

$$A_1 = 0 ; A_5 = 0 ; A_8 = 0$$

Substituting these into (4.19) and (4.20), it follows that the determinant and the trace are given by

$$\begin{aligned} Det &= -C^* A_9 \left( (G_{12}^* + G_{14}^*) \frac{H^*}{K^*} A_6 + (G_{11}^* + G_{13}^*) \right) \\ Trace &= (A_7 - 1) \frac{Y^*}{K^*} - (A_{10} - 1) \frac{C^*}{K^*} \end{aligned}$$

In the case that (4.21) holds, it follows that

$$A_9 = \frac{1}{Z_1} < 0$$

Since  $G_{11}^* + G_{13}^* < 0$ , in order for the determinant to be positive the following condition must hold:

$$A_6 > 0 \tag{4.22}$$

I proceed by showing that if  $A_6 > 0$ , then the trace cannot be negative; thus indeterminacy cannot exist. From (4.20) it follows that in order for the trace to be negative, the following condition must hold

$$(A_7 - 1) Y^* < (A_{10} - 1) C^* \tag{4.23}$$

When (4.21) holds

$$A_{10} = A_6 G_2^* \frac{H^*}{C^*} \quad (4.24)$$

and therefore (4.23) can be written as

$$(G_1^* + G_3^*) K^* + (G_2^* + G_4^*) H^* A_6 - Y^* < A_6 G_2^* H^* - C^* \quad (4.25)$$

Rearranging terms and noticing that

$$Y^* = G_1^* K^* + G_2^* H^*$$

it follows that (4.25) can be written as

$$G_3^* K^* + G_4^* H^* A_6 < G_2^* H^* - C^* \quad (4.26)$$

From (4.1) it follows that

$$C^* - G_2^* H^* = (G_1^* - \delta) K^*$$

and since (4.56) implies

$$G_1^* = \delta + \rho$$

it follows that

$$C^* - G_2^* H^* = \rho K^* > 0$$

Thus, in order for (4.26) to be satisfied, it must be true that

$$G_3^* K^* + G_4^* H^* A_6 < 0$$

Notice that the first term,  $G_3^* K^*$ , is positive. Thus, the trace can be negative only if

$$A_6 < 0$$

However, it was found in (4.22) that in order for the determinant to be positive it must be that

$$A_6 > 0$$

Thus, in the case of no income effect on the demand for leisure, there cannot exist an indeterminate steady state. ■

**Proof.** [Proposition 3] From (4.11) and (4.17), it follows that

$$\eta_{V,H}^* \frac{G_2^*}{H^*} = \frac{V''}{Q'}$$

Notice that

$$\eta_{W,H}^* = \eta_{G_2,H}^* = \frac{H}{G_2^*} [G_{22}^* + G_{24}^*]$$

It then follows immediately that

$$\begin{aligned} \eta_{W,H}^* > \eta_{V,H}^* &\Leftrightarrow G_{22}^* + G_{24}^* > \frac{V''}{Q'} \\ \eta_{W,H}^* < \eta_{V,H}^* &\Leftrightarrow G_{22}^* + G_{24}^* < \frac{V''}{Q'} \end{aligned}$$

■

**Proof.** [Proposition 4] Assume in contradiction that  $\eta_{W,H}^* > \eta_{V,H}^*$ . It then follows from the previous proposition that

$$G_{22}^* + G_{24}^* > \frac{V''}{Q'}$$

The determinant is given by

$$Det = \begin{pmatrix} (G_{12}^* + G_{14}^*) \frac{H^* Y^*}{K^*} (A_5(1 - A_7) + A_6 A_8) + \\ (G_{12}^* + G_{14}^*) \frac{H^* C^*}{K^*} ((A_{10} - 1) A_5 - A_6 A_9) + \\ (G_{11}^* + G_{13}^* (A_8 Y^* - A_9 C^*)) \end{pmatrix}$$

Substitute for

$$A_8 = \frac{1}{Y^*} (H^* (G_2 + G_4) A_5)$$

to get

$$Det = \begin{pmatrix} (G_{12}^* + G_{14}^*) \frac{H^* Y^*}{K^*} A_5 \left( (1 - A_7) + \frac{H^* (G_2 + G_4) A_6}{Y^*} \right) + \\ (G_{12}^* + G_{14}^*) \frac{H^* C^*}{K^*} ((A_{10} - 1) A_5 - A_6 A_9) + \\ (G_{11}^* + G_{13}^* (H^* A_5 (G_2 + G_4) - A_9 C^*)) \end{pmatrix}$$

Substitute for

$$A_6 = \frac{\left( \frac{K^*}{H^*} A_2 \right)}{\left( 1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1 \right)}$$

to get

$$Det = A_{11} + A_{12} + A_{13}$$

where

$$\begin{aligned} A_{11} &= (G_{12}^* + G_{14}^*) \frac{H^* Y^*}{K^*} A_5 \left( (1 - A_7) + \frac{(G_2^* + G_4^*)}{Y^*} \frac{K^* A_2}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right)} \right) \\ A_{12} &= (G_{12}^* + G_{14}^*) \frac{C^*}{K^* \left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right)} \left( (A_{10} - 1) \frac{C^* A_1}{Z_1} - K^* A_9 A_2 \right) \\ A_{13} &= (G_{11}^* + G_{13}^*) (H^* (G_2^* + G_4^*) A_5 - A_9 C^*) \end{aligned}$$

$A_{13}$  can be signed to be negative. To perceive this, note that

$$G_{11}^* + G_{13}^* < 1$$

and that

$$A_9 = \frac{1}{Z_1} (1 - Z_2 A_5) = \frac{1}{Z_1} \left( \frac{1}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}} \right) < 0$$

where  $A_9$  can be signed since  $Q'' > 0$  and  $G_{22}^* + G_{24}^* > \frac{V''}{Q'}$  implies  $A_1 < 0$  and  $A_5 > 0$ . I proceed by analyzing the conditions under which  $A_{11} + A_{12}$  is positive. As will be shown, in the presence of negative income-effects on the demand for leisure  $A_{11} + A_{12}$  is always negative; thus there can be no indeterminacy. Notice that  $\frac{(G_{12}^* + G_{14}^*) H^*}{K^*} > 0$  can be factored out of  $A_{11}$  and  $A_{12}$ . Thus, it is sufficient to look at

$$A_{14} = \left( \begin{array}{c} Y^* A_5 \left( (1 - A_7) + \frac{(G_2^* + G_4^*)}{Y^*} \frac{K^* A_2}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right)} \right) + \\ \frac{C^*}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right)} \left( (A_{10} - 1) \frac{C^* A_1}{H^* Z_1} - \frac{K^* A_9 A_2}{H^*} \right) \end{array} \right)$$

Substitute for

$$A_7 = \frac{1}{Y^*} ((G_1^* + G_3^*) K^* + (G_2^* + G_4^*) H^* A_6)$$

to get

$$A_{14} = \left( \begin{array}{c} A_5 \left( (Y^* - (G_1^* + G_3^*) K^* - (G_2^* + G_4^*) H^* A_6) + (G_2^* + G_4^*) \frac{K^* A_2}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right)} \right) + \\ \frac{C^*}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right)} \left( (A_{10} - 1) \frac{C^* A_1}{H^* Z_1} - \frac{K^* A_9 A_2}{H^*} \right) \end{array} \right)$$

Substitute for

$$A_6 = \frac{A_2 \left(\frac{K^*}{H^*}\right)}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}}$$

to get

$$A_{14} = \left( A_5 (Y^* - (G_1^* + G_3^*) K^*) + \frac{C^*}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right) H^*} \left( (A_{10} - 1) \frac{C^* A_1}{Z_1} - K^* A_9 A_2 \right) \right)$$

Substitute for

$$A_5 = \frac{A_1 \left(\frac{C^*}{H^*}\right) \frac{1}{Z_1}}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}}$$

to get

$$A_{14} = \frac{C^*}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right) H^*} \left( \frac{A_1 (Y^* - (G_1^* + G_3^*) K^*)}{Z_1} + (A_{10} - 1) \frac{C^* A_1}{Z_1} - K^* A_9 A_2 \right)$$

Since  $\frac{C^*}{\left(1 + \frac{Z_2}{Z_1} \frac{C^*}{H^*} A_1\right) H^*} > 0$ , the sign of  $A_{14}$  is determined by the sign of

$$A_{15} = \frac{A_1 (Y^* - (G_1^* + G_3^*) K^*)}{Z_1} + C^* \left( (A_{10} - 1) \frac{C^* A_1}{Z_1} - K^* A_9 A_2 \right)$$

Substitute for

$$A_9 = \frac{1}{Z_1} (1 - Z_2 A_5) ; A_{10} = -\frac{Z_2}{Z_1} A_6$$

$$A_{15} = \frac{1}{Z_1} \left( A_1 (Y^* - (G_1^* + G_3^*) K^*) - C^* \left( \left( \frac{Z_2}{Z_1} A_6 + 1 \right) C^* A_1 + K^* (1 - Z_2 A_5) A_2 \right) \right)$$

Since  $Z_1 < 0$ , in order for  $A_{15}$  to be positive,  $A_{16}$  must be negative, where  $A_{16}$  is given by

$$A_{16} = A_1 (Y^* - (G_1^* + G_3^*) K^*) - C^* \left( \left( \frac{Z_2}{Z_1} A_6 + 1 \right) A_1 + K^* (1 - Z_2 A_5) A_2 \right)$$

Rewrite  $A_{16}$  as

$$A_{16} = A_1 \left( Y^* - (G_1^* + G_3^*) K^* - \left( \frac{Z_2}{Z_1} A_6 + 1 \right) C^* \right) - (K^* (1 - Z_2 A_5) A_2)$$

and, using the fact that  $Y^* - C^* = \delta K^*$ ,  $A_{16}$  can be written as

$$A_{16} = A_{17} - A_{18}$$

where

$$\begin{aligned} A_{17} &= A_1 \left( (\delta - G_1^*) K^* - G_3^* K^* - \frac{Z_2}{Z_1} A_6 C^* \right) \\ A_{18} &= K^* A_2 (1 - Z_2 A_5) \end{aligned}$$



$A_{17}$  can be signed to be positive in the following way: recall that

$$A_2 = \frac{(G_{21}^* + G_{23}^*)}{\left(\frac{V''(H^*)}{Q'(C^*)} - (G_{22}^* + G_{24}^*)\right)} ; A_6 = \frac{A_2 \left(\frac{K^*}{H^*}\right)}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}}$$

At the beginning of the proof, it was assumed in contradiction that

$$\frac{V''(H^*)}{Q'(C^*)} - (G_{22}^* + G_{24}^*) < 0$$

and therefore

$$A_2 < 0$$

and thus

$$A_6 < 0$$

Since  $Z_2 > 0$  and  $Z_1 < 0$ , it follows that

$$-\frac{Z_2}{Z_1} A_6 C^* < 0$$

Moreover, in the steady state

$$G_1^* = \rho + \delta$$

and since  $G_3 \geq 0$  and  $A_1 < 0$ , it follows that  $A_{17}$  is positive. The sign of  $A_{18}$  can be determined as follows: by substituting for  $A_5$ ,  $A_{18}$  can be written as

$$A_{18} = K^* A_2 \left( 1 - Z_2 \left( \frac{A_1 \left(\frac{C^*}{H^*}\right) \frac{1}{Z_1}}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}} \right) \right)$$

and thus

$$A_{18} = \frac{K^* A_2}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}}$$

Since  $A_2 < 0$ , it follows that  $A_{18} < 0$ , and therefore

$$A_{16} = A_{17} - A_{18} > 0$$

However, recall that it was shown that in order for the determinant to be positive,  $A_{16}$  had to be negative. Thus, the proof is completed. ■

## Appendix C:

### Deriving the Dynamic System from Section 2

The next set of differential equations is obtained

$$\dot{\lambda}_t = [\rho + \delta - r_t] = \rho + \delta - qe^{y_t - k_t} \quad (4.27)$$

$$\dot{k}_t = e^{y_t - k_t} - e^{c_t - k_t} - \delta \quad (4.28)$$

In order to solve the model, the two dynamic equations (4.27) and (4.28) need to be transformed into a pair of autonomous differential equations. Thus,  $y_t - k_t$  and  $c_t - k_t$  need to be expressed as functions of  $\lambda_t$  and  $k_t$ . By log-linearizing equations (2.3) and (2.15), it follows that

$$\hat{\lambda}_t = \frac{-C^*}{\left(C^* - \frac{H^{*1+\chi}}{1+\chi}\right)} \hat{c}_t + \frac{H^{*1+\chi}}{\left(C^* - \frac{H^{*1+\chi}}{1+\chi}\right)} \hat{h}_t \quad (4.29)$$

$$\hat{h}_t = \left(\frac{1}{1+x}\right) \hat{y}_t \quad (4.30)$$

Equation (4.30) implies

$$\hat{h}_t = \frac{\alpha}{(1+\chi-\beta)} \hat{k}_t \quad (4.31)$$

From equations (4.30) and (4.31), it follows that  $(\hat{y}_t - \hat{k}_t)$  can be expressed as a function of  $\hat{k}_t$

$$\hat{y}_t - \hat{k}_t = \left(\frac{(\alpha-1)(1+\chi)+\beta}{1+\chi-\beta}\right) \hat{k}_t \quad (4.32)$$

Similarly,  $(\hat{c}_t - \hat{k}_t)$  can be expressed as a function of  $\hat{k}_t$  and  $\hat{\lambda}_t$

$$\hat{c}_t - \hat{k}_t = -\left(1 - \frac{H^{*1+\chi}}{C^*(1+\chi)}\right) \hat{\lambda}_t - \left(1 + \frac{\alpha H^{*1+\chi}}{C^*[1+\chi-\beta]}\right) \hat{k}_t \quad (4.33)$$

Equations (4.32) and (4.33) imply that the two differential equations (4.27) and (4.28) can be rewritten as

$$\dot{k} = e^{\left(\frac{(\alpha-1)(1+\chi)+\beta}{1+\chi-\beta}\right)\hat{k}+(y^*-k^*)} - e^{-\left(1-\frac{H^{*1+\chi}}{C^*(1+\chi)}\right)\hat{\lambda}_t - \left(\frac{\alpha H^{*1+\chi}}{C^*(1+\chi-\beta)}+1\right)\hat{k}_t+(c^*-k^*)} - \delta \quad (4.34)$$

$$\dot{\lambda} = \rho + \delta - qe^{\left(\frac{(\alpha-1)(1+\chi)+\beta}{1+\chi-\beta}\right)\hat{k}+(y^*-k^*)} \quad (4.35)$$

Any trajectory that solves (4.34) and (4.35), subject to the transversality condition (2.7) and the initial condition  $k_0$ , is an equilibrium path.

## Deriving the Dynamic System from Appendix A

The two dynamic equations (4.1) and (4.10) can be written as

$$\dot{k}_t = e^{y_t - k_t} - e^{c_t - k_t} - \delta \quad (4.36)$$

$$\dot{\lambda}_t = (\rho + \delta - r_t) \quad (4.37)$$

In order for (4.36) and (4.37) to be expressed as an autonomous pair of differential equations,  $y_t - k_t$  and  $c_t - k_t$  need to be expressed as functions of  $\lambda_t$  and  $k_t$ .

**Deriving  $h_t$  as a Function of  $k$  and  $\lambda$**  By substituting (4.17) in (4.11), it follows that

$$V'(e^{h_t}) = Q'(e^{c_t})G_2(e^{k_t}, e^{h_t}, e^{\bar{k}_t}, e^{\bar{h}_t}) \quad (4.38)$$

and log-linearizing (4.38) it follows that

$$\hat{h}_t = A_1 \left( \frac{C^*}{H^*} \right) \hat{c}_t + A_2 \left( \frac{K^*}{H^*} \right) \hat{k}_t \quad (4.39)$$

The constants  $A_1$  and  $A_2$  are given by

$$A_1 = \frac{G_2(K^*, H^*, \bar{K}^*, \bar{H}^*)Q''(C^*)}{Q'(C^*)A_3} \quad (4.40)$$

$$A_2 = \frac{A_4}{A_3} \quad (4.41)$$

$$A_3 = \frac{V''(H^*)}{Q'(C^*)} - \left( G_{22}(K^*, H^*, \bar{K}^*, \bar{H}^*) + G_{24}(K^*, H^*, \bar{K}^*, \bar{H}^*) \right) \quad (4.42)$$

$$A_4 = G_{21}(K^*, H^*, \bar{K}^*, \bar{H}^*) + G_{23}(K^*, H^*, \bar{K}^*, \bar{H}^*) \quad (4.43)$$

By log-linearizing (4.8) it follows that

$$\hat{c}_t = \frac{1}{Z_1} \hat{\lambda}_t - \frac{Z_2}{Z_1} \hat{h}_t \quad (4.44)$$

where the constants  $Z_1$  and  $Z_2$  are given by<sup>39</sup>

$$Z_1 = \left( \frac{U''}{U'} Q' + \frac{Q''}{Q'} \right) C^*$$

$$Z_2 = -\frac{U''}{U'} V' H^*$$

---

<sup>39</sup>In order to ease the notational burden, I denote

$$U' = U'(Q(C_t^*) - V(H_t^*)).$$

A similar notation is used for the remaining expressions whenever there is no risk of confusion.

Note that (4.3) to (4.5) imply that  $Z_1 < 0$ . Similarly, (4.3), (4.5), and (4.6) imply that  $Z_2 > 0$ .

In order to express  $h_t$  in terms of  $k_t$  and  $\lambda_t$ , I substitute (4.44) into (4.39) to get

$$\widehat{h}_t = A_5 \widehat{\lambda}_t + A_6 \widehat{k}_t \quad (4.45)$$

where the constants  $A_5$  and  $A_6$  are given by

$$A_5 = \frac{A_1 \left(\frac{C^*}{H^*}\right) \frac{1}{Z_1}}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}} \quad (4.46)$$

$$A_6 = \frac{A_2 \left(\frac{K^*}{H^*}\right)}{1 + A_1 \left(\frac{C^*}{H^*}\right) \frac{Z_2}{Z_1}} \quad (4.47)$$

**Deriving  $y_t$  as a Function of  $k$  and  $\lambda$**  By log-linearizing (4.15) and using (4.45), it follows that

$$\widehat{y}_t = A_7 \widehat{k}_t + A_8 \widehat{\lambda}_t \quad (4.48)$$

where the constants  $A_7$  and  $A_8$  are given by

$$A_7 = \frac{1}{Y^*} ((G_1 + G_3) K^* + (G_2 + G_4) H^* A_6) \quad (4.49)$$

$$A_8 = \frac{1}{Y^*} (H^* (G_2 + G_4) A_5) \quad (4.50)$$

**Deriving  $c_t$  as a Function of  $k$  and  $\lambda$ .** Substituting (4.45) in (4.44) gives

$$\widehat{c}_t = A_9 \widehat{\lambda}_t + A_{10} \widehat{k}_t \quad (4.51)$$

where the constants  $A_9$  and  $A_{10}$  are given by

$$A_9 = \frac{1}{Z_1} (1 - Z_2 A_5) \quad (4.52)$$

$$A_{10} = -\frac{Z_2}{Z_1} A_6 \quad (4.53)$$

**The Autonomous Pair of Differential Equations** Given (4.45), (4.48), and (4.51), (4.36) and (4.37) can be expressed as follows

$$\dot{\lambda}_t = \rho + \delta - G_1 \left( e^{\widehat{k}_t + k^*}, e^{\widehat{h}_t + h^*}, e^{\widehat{k}_t + k^*}, e^{\widehat{h}_t + h^*} \right) \quad (4.54)$$

$$\dot{k}_t = e^{(A_7 - 1)\widehat{k}_t + A_8 \widehat{\lambda}_t + (y^* - k^*)} - \delta - e^{A_9 \widehat{\lambda}_t + (A_{10} - 1)\widehat{k}_t + (c^* - k^*)} \quad (4.55)$$

Note that in the steady state

$$G_1^* = \rho + \delta \quad (4.56)$$





Figure 3: Response of Hours at the First Period - Income Effect

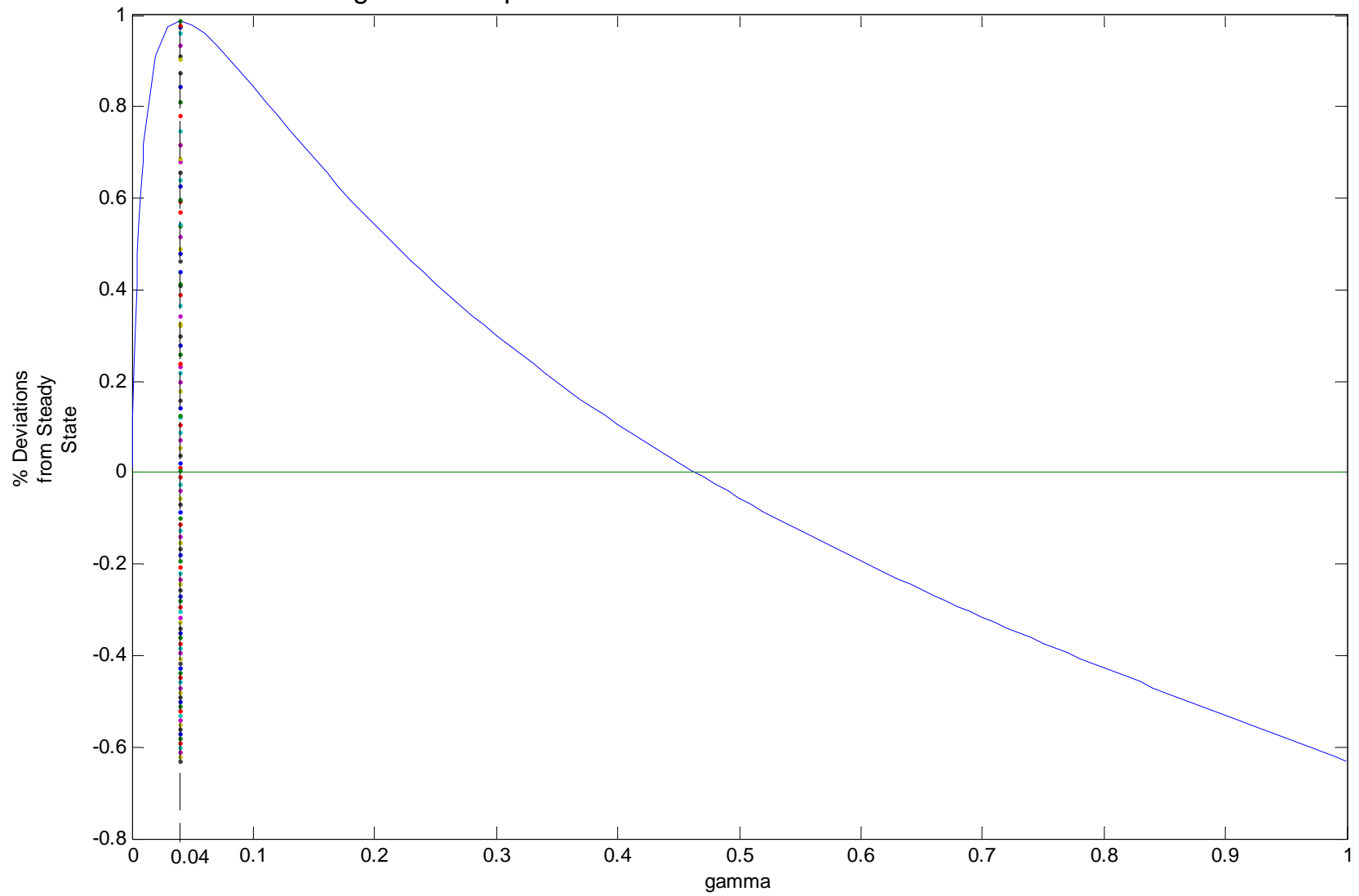


Figure 4: Area of Indeterminacy

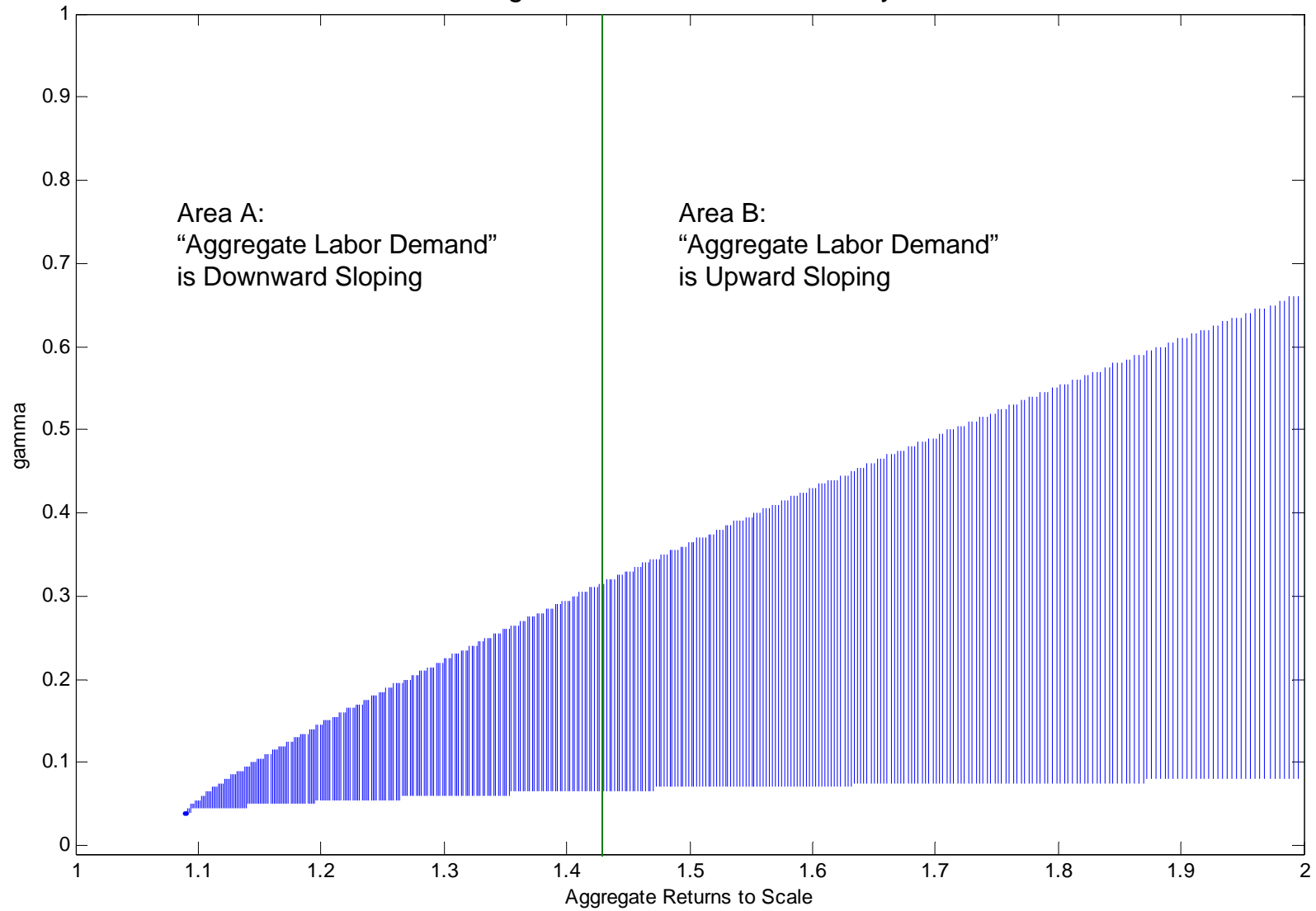




Figure 5: Value of Coefficients

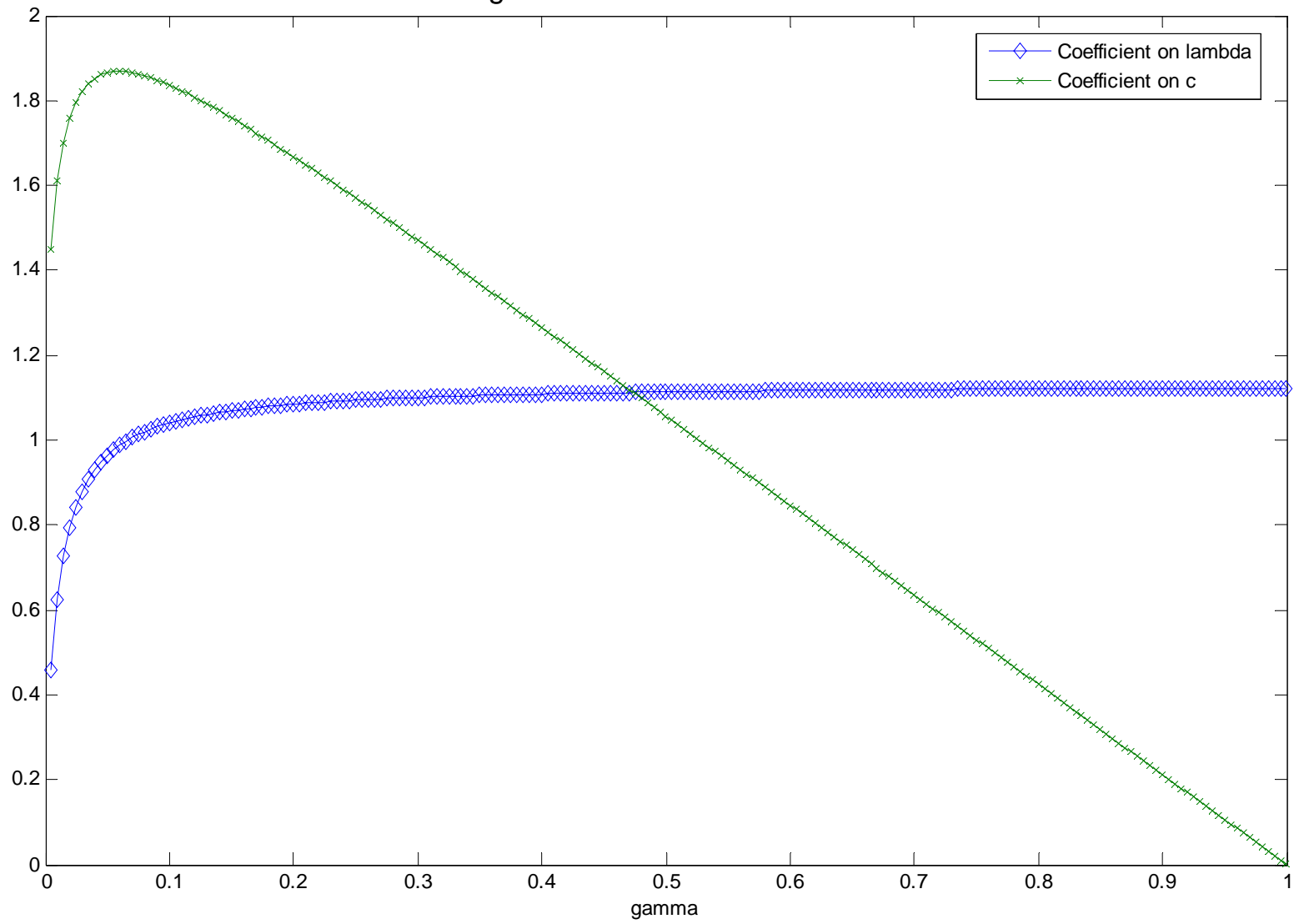


Figure 6: Response of Hours at First Period - Income Effect

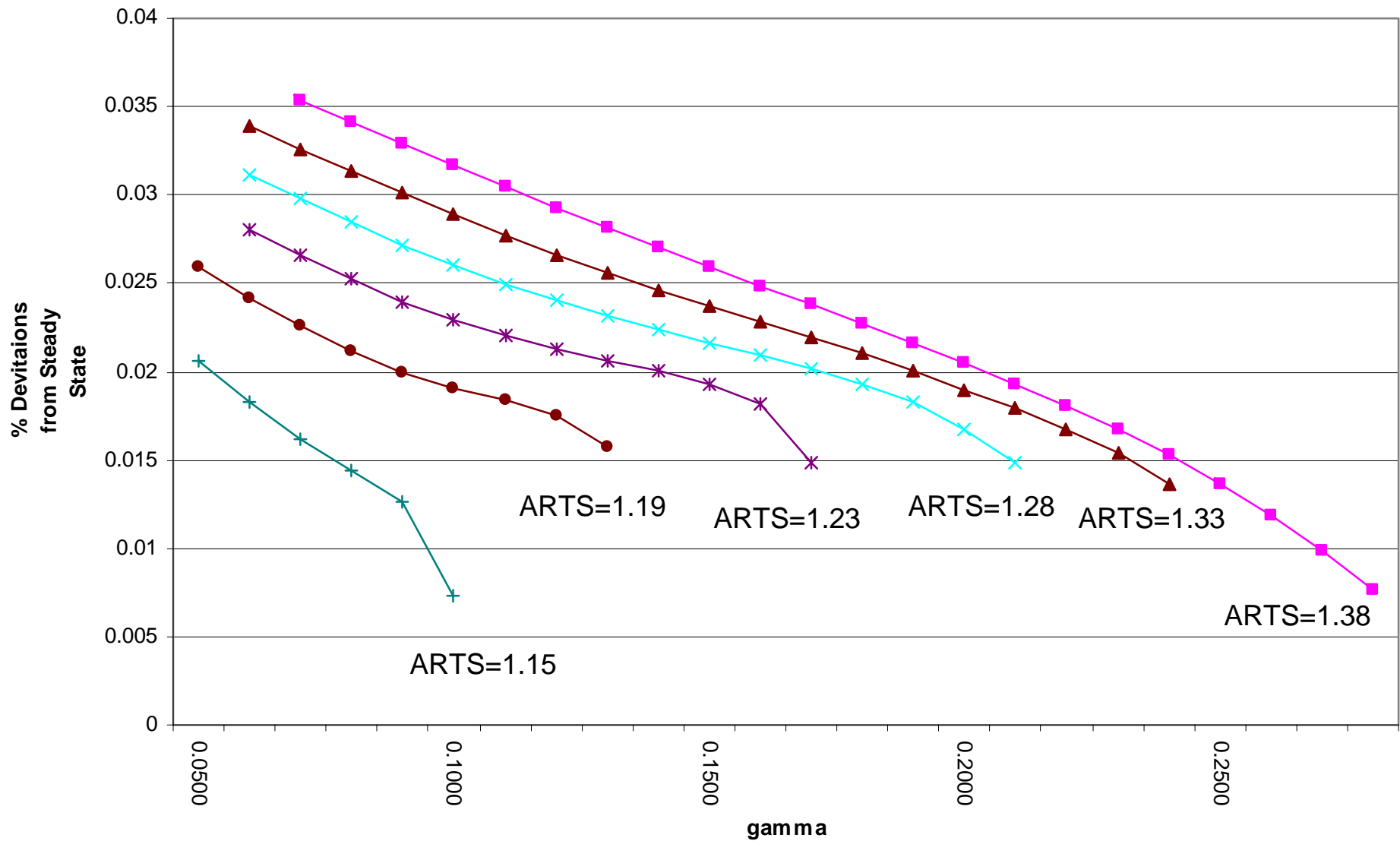


Figure 7: STD(H)/STD(Y)

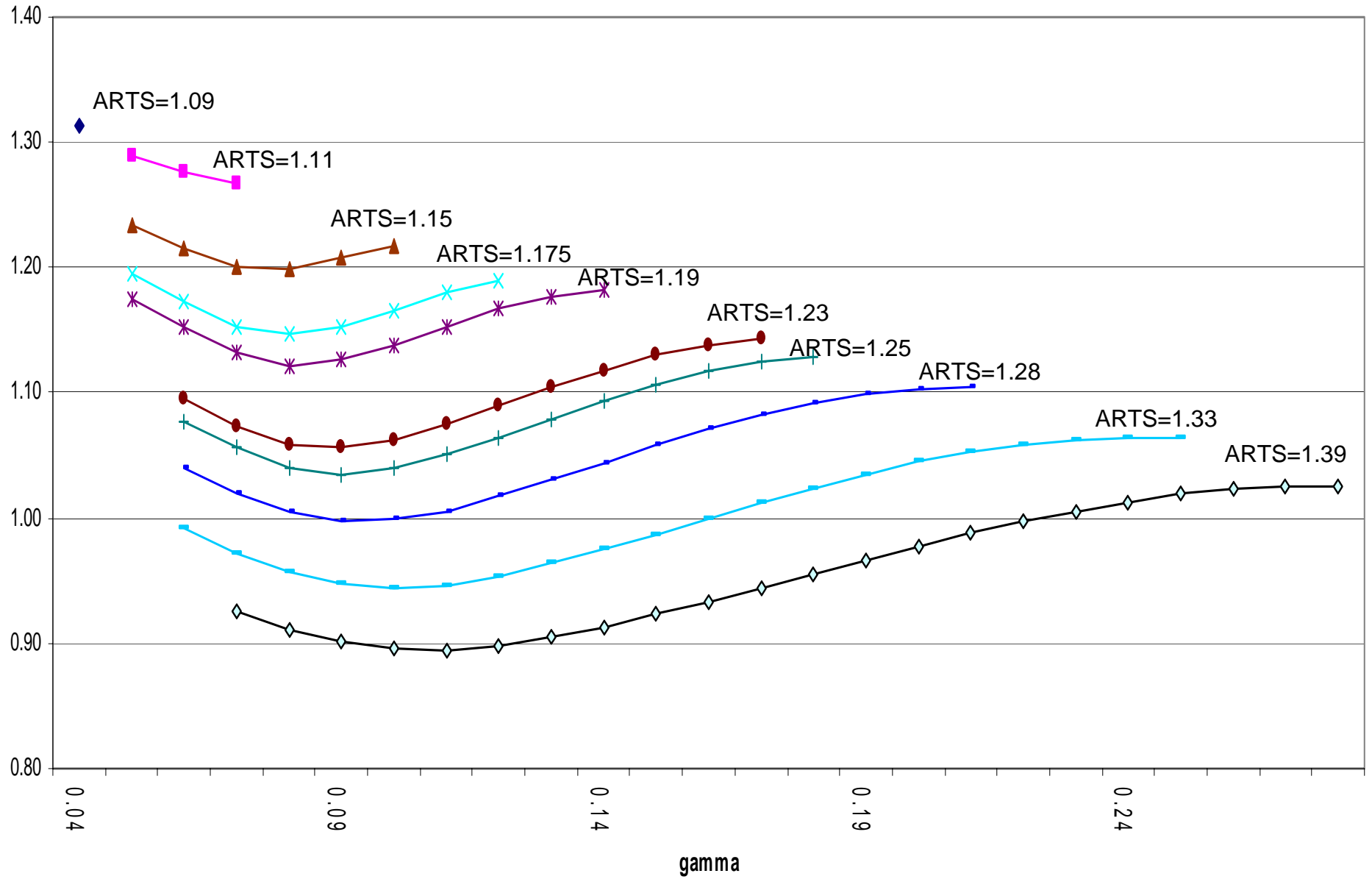




Figure 9: STD(I)/STD(Y)

