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in Dynamic Settings**

By  
B. Douglas Bernheim  
Stanford University  
And  
Sita Nataraj Slavov  
Stanford University

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Stanford Institute for Economic Policy Research  
Stanford University  
Stanford, CA 94305  
(650) 725-1874

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# A Solution Concept for Majority Rule in Dynamic Settings\*

B. Douglas Bernheim  
Department of Economics  
Stanford University  
and

National Bureau of Economic Research  
bernheim@stanford.edu

Sita Nataraj Slavov  
Department of Economics  
Occidental College  
sslavov@oxy.edu

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## Abstract

We define and explore the notion of a Dynamic Condorcet Winner (DCW), which extends the notion of a Condorcet winner to dynamic settings. We show that, for every DCW, every member of a large class of dynamic majoritarian games has an equivalent equilibrium, and that other equilibria are not similarly portable across this class of games. Existence of DCWs is guaranteed when members of the community are sufficiently patient. We characterize sustainable and unsustainable outcomes, study the effects of changes in the discount factor, investigate efficiency properties, and explore the potential for achieving renegotiation-proof outcomes. We apply this solution concept to a standard one-dimensional choice problem wherein agents have single-peaked preferences, as well as to one involving the division of a fixed aggregate payoff.

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## 1 Introduction

A central objective of Political Economy is to understand collective choice with majoritarian institutions. For such institutions, political economists often attempt to infer plausible outcomes directly from the majority preference relation. The best known and most widely used solution concept of this type is the notion of a Condorcet winner, defined as a policy that majority-defeats all other policies.<sup>1</sup>

When Condorcet winners exist, they emerge as equilibrium outcomes in a large and important class of game forms, including models of electoral competition (Downs [1957]), representative democracy (Besley and Coate [1997]), and agenda-setting (Ferejohn, Fiorina, McKelvey, [1987], and Bernheim, Rangel, and Rayo [2006]). Therefore, even when one is not entirely certain of the best way to model the details of a complex majoritarian institution, it is still often reasonable to assume that the institution can implement Condorcet winners.<sup>2</sup> Unfortunately, the existence of Condorcet winners is frequently problematic. The concept suffers from a curse of dimensionality: with a sufficiently rich policy space, it is virtually certain that a Condorcet winner does not exist.<sup>3</sup>

In addition, the notion of a Condorcet winner is not an appropriate solution concept for dynamic collective choice problems. This is an important limitation because dynamic issues are central to numerous public policies issues, such as the design of social security, the use of public debt, the configuration of an optimal tax system, the management of monetary policy, and the provision of incentives for investment and R&D. In recent years, the performance of majoritarian institutions in these dynamic settings has attracted growing interest and attention.<sup>4</sup>

The purpose of the current paper is to propose and develop a solution concept

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<sup>1</sup>Other examples include the Condorcet set (Miller [1977]), defined as the smallest set such that no element of the set is majority defeated by an element outside the set, and the uncovered set (Miller [1980], McKelvey [1986]), which consists of policies  $x$  such that the set of policies majority defeated by  $x$  is not strictly contained in the set of policies majority defeated by any other policy  $y$ .

<sup>2</sup>McKelvey [1986] makes a similar observation concerning the uncovered set. There is, of course, no guarantee that Condorcet winners are the most plausible equilibrium outcome for any particular majoritarian game.

<sup>3</sup>See, e.g., Plott [1967], Rubinstein [1979], Schofield [1983], Cox [1984], Le Breton [1987], Banks [1995], and Saari [1997].

<sup>4</sup>Pertinent studies include Baron [1996], Barron and Herron [1999], Krusell and Rios-Rull [1999, 1996], Epple and Riordan [1987], Krusell, Quadrini and Rios-Rull [1997], Glomm and Ravikumar [1992], Perotti [1993], Persson and Tabellini [1994], Alesina and Rodrik [1994], Cooley and Soares [1999], Grossman and Helpman [1998], Rangel [2003], and Banks and Duggan [2000], among others.

that extends the notion of a Condorcet winner to dynamic settings. We summarize collective choices through a *policy program*, which specifies the current policy choice as a function of previous policies, and thereby allows for the possibility that collective choices are history-dependent. A policy program is a Dynamic Condorcet Winner (DCW) if, in all periods and for all histories, the prescribed choice for the current period majority defeats every alternative, in light of the fact that different choices lead to different prescriptions for future periods. In the current paper, we confine attention to models involving infinite repetitions of a single-period problem, with infinitely-lived agents.<sup>5</sup> A description of these environments and a formal definition of the solution concept appear in Section 2, along with a simple motivating example.

Why study DCWs? As mentioned above, the static Condorcet concept is often justified on the grounds that it identifies outcomes that are equilibria for many majoritarian game forms. As we explain in Section 3, DCWs are justified on precisely the same grounds for dynamic versions of the same game forms. Specifically, for any DCW, every dynamic majoritarian game within this class has an equivalent equilibrium. In contrast, other equilibria do not have the properties that ensure their portability across institutions. Thus, dynamic equilibria corresponding to DCWs are robust with respect to institutional details, while other equilibria are not. This observation provides an argument for studying DCWs rather than dynamic games whenever the political institution of interest is difficult to model (perhaps due to complexity or procedural ambiguities), but one is nevertheless reasonably confident that it could implement Condorcet winners in a static setting.

As we demonstrate, one can usually restrict attention to DCWs that are completely stationary both on and off the equilibrium path (see Section 4.1). This greatly simplifies the task of characterizing DCWs, particularly compared with subgame perfect equilibria for repeated games.<sup>6</sup>

When a Condorcet winner exists for the single-period problem, the policy program that assigns this outcome irrespective of history is a DCW. When agents are sufficiently patient, the community can also achieve any outcome that is part of a Condorcet cycle. Consequently, in contrast to Condorcet winners, DCWs *always* exist when agents are patient. This point, which we develop in Section 4.2., may at first seem surprising: policy programs are extremely high-dimensional objects, and

<sup>5</sup>The concept is easily generalized and extended to other settings such as those with overlapping generations of agents (e.g., Slavov [2006a]).

<sup>6</sup>As shown by Abreu [1986], optimal punishments for repeated games are often non-stationary.

Condorcet winners tend not to exist in high dimensional spaces (recall the references in footnote 4). The key point with respect to existence is that history dependence can break the curse of dimensionality.

In many familiar environments, every outcome is part of a Condorcet cycle (McKelvey [1976, 1979], Schofield [1978]). In combination with the aforementioned result, this observation provides us with a “folk theorem.” We discuss the appropriate interpretation of this result in Section 4.2, and argue that the DCW concept remains useful despite the potential indeterminacy.<sup>7</sup> In particular, we identify conditions under which certain outcomes are unsustainable even when group members are extremely patient (Section 4.3), exhibit assumptions that give rise to those conditions in the context of the median voter model, and demonstrate generally that the set of DCWs shrinks as the group members become more impatient (Sections 4.3 and 4.4). The latter result allows us to associate each DCW with a degree of sustainability, and to identify the most sustainable DCWs.

The efficiency properties of DCWs are also notable (Section 4.5). The efficient frontier of the DCW payoff set always lies on the efficient frontier of the feasible payoff set. Moreover, we show that it is always possible to sustain these efficient outcomes through DCWs that never depart from the efficient frontier of the feasible payoff set for any history. The strategic equilibria corresponding to these *universally efficient* DCWs satisfy a very demanding standard of renegotiation-proofness. Universal efficiency therefore emerges as a natural refinement criterion – one that can potentially resolve the indeterminacy associated with our folk theorem.<sup>8</sup> For example, in the case of the repeated median voter problem, we demonstrate with considerable generality that the repeated static solution is the unique universally efficient outcome, even with extremely patient group members.

In Section 5, we apply our concept to a problem involving pure distribution (division a fixed payoff). In addition to characterizing the DCW set, we draw implications for distributional policy. No alternative is more sustainable than the egalitarian outcome, and movements toward egalitarianism always (weakly) enhance sustainability. Moreover, penalizing one party requires a measure of equality between the

<sup>7</sup>Notably, a folk theorem does not emerge in collective choice problems with overlapping generations, even if agents apply no discounting within their own lifespans. As Slavov [2006a] demonstrates, DCWs yield useful predictions in such contexts.

<sup>8</sup>In contrast, for repeated games, it is difficult to assure the existence of equilibria satisfying reasonable notions of renegotiation-proofness (Bernheim and Ray [1989]; Farrell and Maskin [1989]).

other two. Consequently, political feasibility places upper limits on wealth, but does not place lower limits on poverty.

This paper is perhaps most closely related to work by Krusell and Ríos-Rull [1999], who proposed another dynamic generalization of Condorcet winners, known as a “recursive political equilibrium.”<sup>9</sup> Their concept allows continuation paths to depend on a structural state variable, but not on any other aspect of history.<sup>10</sup> In the absence of a structural state variable (that is, for the set of environments considered in this paper), the concept effectively treats each period as a separate problem, selecting a static Condorcet winner when one exists. Thus, for the environments considered in this paper, recursive political equilibria fail to exist when there is no Condorcet winner for the static problem.

## 2 The Problem and the Solution Concept

### 2.1 The Environment

A finite set of infinitely-lived agents,  $\{1, 2, \dots, N\}$  with  $N \geq 3$ , constitutes a community. Time unfolds in a sequence of discrete periods, indexed  $t = 0, 1, 2, \dots$ . In each period  $t$ , the community must select some policy  $x$  from the set  $X$ . Single period payoffs are given by the mapping  $\eta : X \rightarrow \mathcal{R}^N$ , where  $\eta_i(x)$  denotes the payoff to agent  $i$  from policy  $x$ .

Let  $U$  denote the image of  $X$  under  $\eta$ . Without loss of generality, we can think of the community as selecting  $u$  from the set  $U$ ; henceforth, we therefore refer to each  $u$  as a policy, and usually (but not always) suppress references to  $X$ . We typically assume that  $U$  is compact, convex, or both. These assumptions are satisfied, for example, if  $X$  corresponds to lotteries over some finite set of deterministic policies.

Let  $M$  denote the smallest integer not less than  $\frac{N}{2}$ ; when  $N$  is even,  $M = \frac{N}{2}$ , and when  $N$  is odd,  $M = \frac{N+1}{2}$ . We define a binary relation  $R$  on elements of  $\mathbb{R}^N$ , such that  $w'Rw''$  iff  $\#\{i \in 1, \dots, N \mid w'_i \geq w''_i\} \geq M$  (where  $\#$  denotes the

<sup>9</sup>Our paper is also related, though somewhat less closely, to work by Roberts [2006]. Roberts assumes that, each period, the status quo is compared to a randomly chosen alternative based on majority rule; the winner becomes the status quo in the following period. With limited history dependence (voters’ choices are conditioned on the current status quo), a number of outcomes can be sustained. There are also rough parallels to our concept in the literature on time consistency in dynamic cooperative games (see e.g. Petrosjan and Zenkevich [1996], or Filar and Petrosjan [2000]).

<sup>10</sup>Thus, the relation between our concept and theirs is analogous to the relation between subgame perfect equilibria and Markov perfect equilibria.

cardinality of a set). Viewing  $w'$  and  $w''$  as vectors of scalar payoffs, this is the conventional majority preference relation. Corresponding to  $R$ , there is a strict relation  $P$  defined as follows:  $w'Pw''$  iff  $w'Rw''$  and  $\sim [w''Rw']$ .

A policy sequence  $(u^0, u^1, \dots)$  is feasible iff  $u^t \in U$  for all  $t$ . We assume that each individual discounts future payoffs at the rate  $\delta$ . For any feasible policy sequences  $\gamma$  and  $\rho$ , we say that  $\gamma$  is *weakly majority preferred* to  $\rho$  (alternatively,  $\gamma R^\delta \rho$ ) iff  $[\sum_{t=0}^{\infty} \delta^t \gamma^t] R [\sum_{t=0}^{\infty} \delta^t \rho^t]$  (keep in mind that the superscript on  $\delta$  is an exponent, while the superscripts on  $\gamma$  and  $\rho$  are indexes). For the case of  $\delta = 1$ , we use the overtaking criterion:  $\gamma R^1 \rho$  iff there exists some finite  $T$  such that, for all  $\tau \geq T$ , we have  $[\sum_{t=0}^{\tau} \gamma^t] R [\sum_{t=0}^{\tau} \rho^t]$ . If  $\gamma R^\delta \rho$  and  $\sim [\rho R^\delta \gamma]$ , we say that  $\gamma$  is *strictly majority preferred* to  $\rho$  (alternatively,  $\gamma P^\delta \rho$ ). In most instances, we omit the modifier “weakly,” stating “strictly” where intended.

## 2.2 Dynamic Condorcet Winners

In dynamic models, it is natural to think that today’s outcomes may affect outcomes in the future. This is the case even when there are no structural links across periods (by way of analogy, consider equilibria based on history dependent strategies in repeated games). We use  $h_t = (u^0, u^1, \dots, u^{t-1})$  to denote the history of policies adopted prior to period  $t$  (a *t-history*). Let  $H_t$  denote the set of all feasible  $t$ -histories (that is,  $(u^0, u^1, \dots, u^{t-1})$  such that  $u^s \in U$  for all  $s \in \{0, 1, \dots, t-1\}$ ). A *dynamic policy program* is a mapping  $\mu : \bigcup_{t=0}^{\infty} H_t \rightarrow U$ . For every period  $t$  and every history of policies adopted prior to period  $t$ ,  $\mu$  identifies the next policy to be implemented. By considering policy programs rather than simply infinite sequences of policies, we allow for the possibility that the community’s decisions may be history-dependent.

For any policy program  $\mu$ , we define the function  $C^\mu$  by

$$C^\mu(h) = (\mu(h), \mu(h, \mu(h)), \dots) \text{ for all } h \in \bigcup_{t=0}^{\infty} H_t$$

For any  $t$ -history  $h_t$ ,  $C^\mu(h_t)$  describes the continuation policy sequence generated by applying the policy program  $\mu$ .

We are now prepared to define our solution concept for the dynamic collective choice problem.

**Definition:** A *Dynamic Condorcet Winner (DCW)* is a policy program  $\mu$  such that

$$C^\mu(h_t) R^\delta (u, C^\mu(h_t, u)) \text{ for all } t = 0, 1, \dots, h_t \in H_t, \text{ and } u \in U.$$

In words,  $\mu$  is a DCW if and only if, for all feasible  $t$ -histories, the prescribed policy for the current period,  $\mu(h_t)$ , is a Condorcet winner, in the sense that it majority-defeats all other policies  $u \in U$ . The important feature of this definition is that, when comparing the prescribed policy for the current period to any other possible policy, each agent considers the fact that the current outcome affects the continuation path.

In subsequent sections, we illustrate the implications of our general results by returning repeatedly to a familiar application, commonly known as the “median voter” model, wherein the policy choice is one-dimensional (without loss of generality,  $X = [0, 1]$ ), each community member’s payoff function,  $\eta_i(x)$ , is continuous and strictly concave (hence “single-peaked”), and  $N$  is odd. One can think of  $x \in X$  as denoting the level or characteristic of some public good. We use  $y_i$  to denote the optimal level of  $x$  from the perspective of agent  $i$ . Without loss of generality, we label the agents so that  $y_i$  is weakly increasing in  $i$ . Agent  $M$  is the “median voter.” If the community makes this choice only once,  $y_M$  is the unique Condorcet winner.

### 2.3 Example

The following example illustrates the concept of a DCW and previews some of our results. A community consisting of three agents, indexed  $i = 1, 2, 3$ , repeatedly confronts the problem of selecting some policy  $x$  from the set  $\{A, B, C, D, E\}$ . In any given period, the payoffs to the agents from each of the policies are as follows:

$$\begin{aligned} A & : (2, 3, 4) \\ B & : (3, 4, 2) \\ C & : (4, 2, 3) \\ D & : (1, 1, 5) \\ E & : (0, 0, 0) \end{aligned}$$

Here the  $i$ -th element in each vector is the payoff to individual  $i$ . Focusing on any single period in isolation, we see that  $A$  is majority preferred to  $C$ ,  $C$  is majority preferred to  $B$ , and  $B$  is majority preferred to  $A$ . Moreover,  $A, B$ , and  $C$  are each majority preferred to  $D$  and  $E$ , and  $D$  is majority preferred to  $E$ . There is no Condorcet winner; i.e., there is no policy that is majority preferred to all other

policies in pairwise comparison.  $A, B$ , and  $C$  form a Condorcet cycle, and  $D$  and  $E$  lie “below” the cycle.

Members of this community should *not*, however, focus on any single period in isolation. Because social outcomes are in principle history-dependent, the majority preference relation depends on individuals’ expectations concerning continuation paths.

Consider the following policy program:  $\mu(h_0) = A$ ,  $\mu(h_t, \mu(h_t)) = \mu(h_t)$  and for any  $x \neq \mu(h_t)$ ,

$$\mu(h_t, x) = \begin{cases} C & \text{if } \mu(h_t) = A \\ B & \text{if } \mu(h_t) = C \\ A & \text{if } \mu(h_t) = B \end{cases} .$$

Notice that this policy program yields  $A$  in every period (that is,  $C^\mu(h_0) = (A, A, \dots)$ ).

As long as the individuals are sufficiently patient ( $\delta \geq \frac{1}{2}$ ),  $\mu$  is a DCW. First consider any history  $h_t$  for which the policy program prescribes  $A$  in the next period (that is,  $\mu(h_t) = A$ ). For any such history, individuals will expect the following: (1) the community will select  $A$  in every subsequent period (that is,  $C^\mu(h_t) = (A, A, \dots)$ ); (2) if the community selects some other policy  $x \neq A$  in any period, then it will select  $C$  in every subsequent period (that is,  $C^\mu(A, \dots, A, x) = (C, C, \dots)$ ). Comparing the two paths  $(A, A, \dots)$  and  $(x, C, C, \dots)$ , it is apparent that a majority (individuals 2 and 3) prefers the first to the second as long as individuals are sufficiently patient ( $\delta \geq \frac{1}{2}$ ).<sup>11</sup>

Now consider any history  $h_t$  for which the policy program prescribes  $C$  in the next period (that is,  $\mu(h_t) = C$ ). For any such history, individuals will expect the following: (1) the community will select  $C$  in this and all subsequent periods (that is,  $C^\mu(h_t) = (C, C, \dots)$ ); (2) if the community selects some other policy  $x \neq C$  in the current period, then the community will select  $B$  in every subsequent period (that is,  $C^\mu(h_t, x) = (B, B, \dots)$ ). Again, if individuals are sufficiently patient ( $\delta \geq \frac{1}{2}$ ), a majority (individuals 1 and 2) prefers  $(C, C, \dots)$  to  $(x, B, B, \dots)$ .

Finally, consider any history  $h_t$  for which the policy program prescribes  $B$  in the next period (that is,  $\mu(h_t) = B$ ). For any such history, individuals will expect the following: (1) the community will select  $B$  in this and all subsequent periods, and (2) if the community selects some other policy  $x \neq B$  in the current period, then it

<sup>11</sup>In particular, if all individuals discount future payoffs at the rate  $\delta$ , then for individual 2, the first path yields a payoff of  $\frac{3}{1-\delta}$ , while the second yields  $x_2 + \frac{2\delta}{1-\delta}$  (where  $x_2$  is the second element in  $x$ ). Individual 2 prefers  $(A, A, \dots)$  to  $(x, C, C, \dots)$  for any  $x$  provided  $\delta \geq \frac{1}{2}$ . A similar calculation shows that individual 3 also prefers  $(A, A, \dots)$  to  $(x, C, C, \dots)$  for any  $x$  provided that  $\delta \geq \frac{1}{2}$ .

will select  $A$  in every subsequent period (that is,  $C^\mu(h_t, x) = (A, A, \dots)$ ). Again, if individuals are sufficiently patient ( $\delta \geq \frac{1}{2}$ ), a majority (individuals 1 and 3) prefers  $(B, B, \dots)$  to  $(x, A, A, \dots)$ .

Putting all of these pieces together, we see that  $\mu$  is indeed a DCW. The choice of  $A$  in every period is majority preferred to all alternatives when a deviation from this path results in a permanent shift to  $C$ ; the choice of  $C$  in every period is majority preferred to all alternatives when a deviation from this path results in a permanent shift to  $B$ ; and the choice of  $B$  in every period is majority preferred to all alternatives when a deviation from this path results in a permanent shift to  $A$ . Moreover, if we were to change the initial choice,  $\mu(h_0)$ , to either  $B$  or  $C$ , the policy program would remain a DCW, and it would yield either  $B$  or  $C$  in every period. Ironically, the majority preference cycle that undermines the existence of a Condorcet winner in the static collective choice problem is actually instrumental in constructing a political equilibrium in the dynamic setting. This observation is generalized subsequently in Theorem 3.

Although it is possible for the community to select  $A$ ,  $B$ , or  $C$ , the community cannot select either  $D$  nor  $E$  in every period.<sup>12</sup> For the purposes of the current illustration, we confine attention to stationary paths. However, our formal analysis also allows for non-stationary outcomes (see in particular Theorem 1). First consider a path along which the community selects  $E$  in every every period. Note that any other continuation path  $(x, y, y, \dots)$  is majority preferred to  $(E, E, \dots)$ . Consequently, regardless of how the continuation path is chosen, a majority would favor the choice  $x \neq E$  to  $E$  in the current period. Now consider a path along which the community selects  $D$  in every period. Note that any other continuation path  $(x, y, y, \dots)$  with  $x \in \{A, B, C\}$  is majority preferred to  $(D, D, \dots)$  unless  $y = E$ . But we have already established that  $(E, E, \dots)$  is not sustainable. Hence  $(D, D, \dots)$  cannot emerge either. This observation is generalized subsequently in Theorem 4.

### 3 Why Study DCWs?

Dynamic Condorcet Winners, like their static counterparts, are non-game-theoretic solutions. They do not include any notion of individual strategies or deviations.

<sup>12</sup>In contrast, if individuals are sufficiently patient, paths along which the community selects either  $D$  or  $E$  in every period are subgame-perfect equilibrium outcomes of a strategic model with two-party Downsian competition in each period (even when all members always vote as if they are pivotal).

There is also no natural analog of “Nash reversion” (constructing punishments based on repetitions of a static Nash equilibrium). Attempting to sustain an equilibrium through “Condorcet reversion” (constructing punishments based on repetitions of a static Condorcet winner) is pointless, as a majority would always prefer the punishment path to the equilibrium continuation path.

Nevertheless, the main value of the DCW concept lies in its relationship to game theoretic solutions. The static Condorcet concept is often justified on the grounds that it identifies outcomes that are equilibria for many majoritarian game forms. As we explain in this section, DCWs are justified on precisely the same grounds for dynamic versions of the same game forms. Specifically, for any DCW, every member of this large class of dynamic political institutions has an equivalent equilibrium. Naturally, like many static institutions that implement Condorcet winners, a dynamic institution within the class of interest may have other equilibria that do not correspond to DCWs. However, none of those other equilibria have the properties that ensure portability across the entire class institutions. Thus, dynamic equilibria corresponding to DCWs are robust with respect to institutional details, while other equilibria are not. These observations provide an argument for studying DCWs rather than dynamic games whenever the political institution of interest is difficult to model (perhaps due to complexity or procedural ambiguities), but one is nevertheless reasonably confident that it could implement Condorcet winners in a static setting.

### 3.1 Downsian Competition

For concreteness, we begin by illustrating the relationship between the DCW concept and repeated Downsian competition with two political parties. However, none of our conclusions are specific to the Downsian model; they apply equally to any static institution for which the equilibrium set always contains all Condorcet winners (if any exist).

In the static Downsian model, there are  $N$  citizens and two political parties,  $L$  and  $R$ . The game consists of two stages. In the first stage, each party proposes a policy  $x_L, x_R \in X$ . Since strategic considerations potentially inject uncertainty either endogenously (through the selection of mixed strategies) or exogenously (e.g., through random tie-breaking), we take  $X$  from the outset to consist of lotteries over some set of deterministic policies. In the second stage, individuals vote for a

party. Thus, a strategy for citizen  $i$  is a mapping from pairs of policy choices into votes. Citizens care only about the policy outcome, and candidates care only about electoral victory (from which they receive a fix payoff).

In equilibrium, we require as-if-pivotal voting behavior, which guarantees that citizens cast their votes for their preferred outcomes, and subgame perfection, which guarantees that they do so for any pair of announced policy platforms. As is well known, if there is a Condorcet winner  $x \in X$ , then there is an equilibrium in which both parties propose  $x$ . If there is no Condorcet winner, then no pure strategy equilibrium exists; however, there may be mixed strategy equilibria.

Now suppose the Downsian game is repeated infinitely. We assume that parties as well as citizens are infinitely-lived, and that all players discount future payoffs at the rate  $\delta$ . At any point during the game, players can condition their choices upon the entire history of play. This history includes, for each period, the policy outcome, both parties' proposals, and the identity of the winning party. Let  $h_t$  denote the citizen-payoff history  $(u_1, \dots, u_{t-1})$  as before, and let  $q_t$  denote all other historical information.

Consider any DCW,  $\mu$ . We argue that there is an equivalent equilibrium for the repeated game. Define  $D = (1, \delta, \delta^2, \dots)'$  and  $\pi(h_t) = C^\mu(h_t)D$ . That is,  $\pi^\mu(h_t)$  is the vector of discounted payoffs from the continuation path  $C^\mu(h_t)$ . For each history  $h_t$ , let  $x^\mu(h_t)$  be a policy for which  $\eta(x^\mu(h_t)) = \mu(h_t)$ . Now construct strategies for the game as follows:

1. If the history is  $(h_t, q_t)$ , both parties propose  $x^\mu(h_t)$ .
2. If the history is  $(h_t, q_t)$ , and if the parties have proposed  $x_L$  and  $x_R$  in the current period, individual  $i$  votes for the party,  $k$ , that proposed the policy yielding the largest value of  $\eta_i(x_k) + \delta\pi_i^\mu(h_t, \eta(x_k))$ . If those values are equal, individual  $i$  votes as follows: (i) if neither party has proposed  $x^\mu(h_t)$ , or if both have proposed  $x^\mu(h_t)$ , individual  $i$  votes for each party with probability  $\frac{1}{2}$ ; (ii) if one and only one party has proposed  $x^\mu(h_t)$ , individual  $i$  votes for that party.

Let's verify that these strategies constitute a subgame perfect equilibrium with as-if pivotal voting. For the usual reasons, it is sufficient to check deviations that are limited to a single period. Suppose the history is  $(h_t, q_t)$  as of period  $t$ . Party  $k$  expects party  $j$  to propose  $x^\mu(h_t)$ . Because  $\mu$  is a DCW, we have

$C^\mu(h_t) R^\delta(u, C^\mu(h_t, u))$  for all  $u \in U$ . Consequently, for every  $x \neq x^\mu(h_t)$ , there are at least  $M$  individuals for whom

$$\eta_i(x^\mu(h_t)) + \delta\pi_i^\mu(h_t, \eta(x^\mu(h_t))) \geq \eta_i(x) + \delta\pi_i^\mu(h_t, \eta(x))$$

According to the voting strategies, all of those individuals will vote for a party proposing  $x^\mu(h_t)$  over a party proposing  $x$ . Therefore, a deviating party will lose with certainty in the current period (instead of winning with probability  $\frac{1}{2}$ ). Regardless of whether it deviates in the current period, the continuation equilibrium implies that the party will win with probability  $\frac{1}{2}$  in every subsequent period. Thus, a deviation makes the party worse off. Now consider the voters. Given the strategies, if a party wins with a platform  $x$ , voter  $i$ 's discounted payoff will be  $\eta_i(x) + \delta\pi_i^\mu(h_t, \eta(x))$ . Thus, voters are behaving optimally, on the assumption that they are always pivotal.

### 3.2 Portability

Economists and political theorists study a variety of static collective choice games for which the equilibrium set always contains all Condorcet winners (if any exist). Aside from the Downsian game, examples include models of representative democracy (Besley and Coate [1999]), and of sequential, comprehensive agendas (Ferejohn, Fiorina, McKelvey, [1987], and Bernheim, Rangel, and Rayo [2006]). Consider the infinitely repeated version of any member of this class of games. Reasoning as in the previous section, one can show that, for any DCW, there is an equivalent equilibrium for the infinitely repeated game. Therefore, DCWs correspond to equilibria that are portable across an important class of dynamic collective choice games.

However, for many equilibria of an infinitely repeated collective choice games belonging to the class described in the previous paragraph, equivalent DCWs do not exist. As an illustration, consider a Downsian model with three voters ( $N = 3$ ). Assume that the utility possibility set is  $U = \{(u, u, u) \mid 0 \leq u \leq 1\}$ . Note that the utility possibility set is Condorcet ranked;  $(1, 1, 1)$  is the unanimously preferred outcome. Yet if  $\delta \geq \frac{1}{2}$ , one can construct subgame-perfect equilibria with as-if-pivotal voting wherein, on the equilibrium path, both parties always propose the worst possible outcome,  $(0, 0, 0)$ .<sup>13</sup>

<sup>13</sup>In fact, Duggan and Fey [2006] show that for  $\delta > \frac{1}{2}$ , any sequence of policies is sustainable under repeated Downsian competition.

We construct such an equilibrium using the following strategies: if there have been no deviations, both parties propose  $(0, 0, 0)$ . If  $R$  proposes any other policy and loses, both parties propose  $(1, 1, 1)$  in every subsequent period, regardless of whether there have been subsequent deviations. In case of any other deviation, both parties continue to propose  $(0, 0, 0)$  in every period. Individuals vote for the party proposing the platform they prefer (accounting for the continuation outcome); if they are indifferent, they vote for  $L$ .

Let's verify that this is a subgame perfect equilibrium with as-if pivotal voting. Since  $L$  wins with certainty in every period on the equilibrium path of every subgame,  $L$  has no incentive to deviate. By construction, voters behave optimally on the assumption that they are always pivotal. Thus, we only need to check  $R$ 's incentives to deviate. We distinguish between two classes of  $t$ -histories: those for which the parties' strategies prescribe  $(0, 0, 0)$  ("type 0 histories") and those for which their strategies prescribe  $(1, 1, 1)$  ("type 1 histories"). Once again, it is sufficient to check deviations that are limited to a single period. Consider any type 0 history. If  $R$  deviates to  $(u, u, u)$  with  $u > 0$ , each individual receives a discounted payoff of  $u$  if  $R$  wins (since both parties revert to  $(0, 0, 0)$  in subsequent periods) and  $\frac{\delta}{1-\delta}$  if  $R$  loses (since both parties will subsequently propose  $(1, 1, 1)$ ). Thus, as long as  $\delta > \frac{1}{2}$ , no one would vote for  $R$ 's alternative proposal  $u$ . Since  $R$  loses in every period whether or not it deviates in the current period, it has no incentive to deviate. Next consider any type 1 history. If  $R$  deviates to  $(u, u, u)$  with  $u < 1$ , each individual receives a discounted payoff of  $u + \frac{\delta}{1+\delta}$  if  $R$  wins and  $1 + \frac{\delta}{1-\delta}$  if  $R$  loses (since both parties will subsequently propose  $(1, 1, 1)$  in every subsequent period regardless of choices in the current period). Thus, no one would vote for  $R$ 's proposal. Since  $R$  loses in every period whether or not it deviates in the current period, it again has no incentive to deviate.

In contrast, no DCW can produce infinite repetitions of the worst outcome,  $(0, 0, 0)$ . Why not? A collective deviation to *any* other policy would make a majority better off *regardless* of the continuation outcome. In fact, the only outcome sustainable as a DCW involves infinite repetitions of the best outcome,  $(1, 1, 1)$  (this result is a consequence of Theorem 4, below).

At the outset of this section, we claimed that equilibria with equivalent DCWs are portable across a large class of important institutions. This portability results from two properties that other equilibria do not share. First, the continuation pol-

icy path for an equilibrium with an equivalent DCW depends only on past policies, and not on any other aspect of the game’s history, including details of the process by which those outcomes were selected. A critical feature of the last equilibrium described above (in which both parties propose the worst possible policy for all voters) is that the continuation policy path depends on aspects of the history other than past policies (specifically, the platform of the losing party).<sup>14</sup> If, in an equilibrium for one institution, the continuation policy path depends on some choice (e.g., the platform of a political party) that has no counterpart in a second institution (e.g., one without political parties), then – unlike DCWs – that equilibrium is not portable to the second institution.

Second, if an equilibrium has an equivalent DCW, then, accounting for equilibrium continuation payoffs, there is always (for every period and every feasible history) an unambiguous Condorcet winner in  $X$ . This property is also critical for portability across institutions: by definition, any institution within the class of interest is capable of selecting that same Condorcet winner for the same history, given the same contingent continuation policy paths. For many institutions, there are also equilibria in which, for some histories, there is no Condorcet winner in  $X$  (accounting for continuation payoffs). However, these equilibria are not generally portable across institutions. To see why, consider an institution, an equilibrium, and a history for which there is no Condorcet winner in  $X$  (accounting for continuation payoffs).<sup>15</sup> For that history, the institution produces some selection from  $X$ . If, with the same conditional continuation policy paths, another institution would produce a different selection from  $X$ ,<sup>16</sup> then – unlike DCWs – that equilibrium is not portable to the second institution.

Thus, DCWs are, in effect, the robust solutions to a large and important class of dynamic majoritarian games. Consequently, one can also think about the DCW set as embodying an equilibrium refinement for such games. In some specific instances,

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<sup>14</sup>The same statement holds for any equilibrium producing an outcome other than  $(1, 1, 1)$ . Indeed, in demonstrating that it is possible to sustain any sequence of policies under repeated Downsian competition when  $\delta > \frac{1}{2}$ , Duggan and Fey [2006] rely on strategies that condition choices on aspects of the game’s history other than policy outcomes.

<sup>15</sup>To avoid conflating this property with the previous one, assume that the equilibrium conditions the continuation policy path only on past policies.

<sup>16</sup>For example, models of legislatures with real-time agenda setting and evolving defaults select Condorcet winners when they exist (provided a sufficient number of legislators have opportunities to make proposals), but when no Condorcet winner exists the outcome depends on the order in which legislators are allowed to make proposals (see Bernheim, Rangel, and Rayo [2006]).

this refinement rules out equilibria which seem implausible (e.g., for the repeated Downsian game, the one in which both parties propose the worst policy for all voters). We do not mean to suggest, however, that an equilibrium is necessarily unreasonable when it corresponds to no DCW. On the contrary, it is entirely possible that a particular institution might give rise to other plausible equilibria. If one is certain that one has the correct model of the pertinent institution, it would be inappropriate to discard such an equilibrium in favor of DCWs. However, if the institution of interest is difficult to model (perhaps due to complexity or procedural ambiguities), but one is nevertheless reasonably confident that it could implement Condorcet winners (when they exist) in a static setting, then DCWs are attractive because they correspond to valid equilibria regardless of institutional details.

## 4 General Properties

### 4.1 Stationary vs. Non-stationary DCWs

Potentially, a policy program is an extremely complicated object. This complexity could in principle reduce the value of the DCW concept by making it difficult to establish general properties, or to apply the concept to particular problems. Fortunately, DCWs turn out to be much simpler to analyze than subgame perfect equilibria. In particular, it is possible in many cases to restrict attention to DCWs that satisfy a strong stationarity property:

**Definition:** A policy program  $\mu$  is *stationary* if, for every  $h_t$ ,  $\mu(h_t) = \mu(h_t, \mu(h_t))$ .

To understand this definition, imagine that the community finds itself in period  $t$  with some history of policies  $h_t$ . The policy program  $\mu$  prescribes some current choice  $u = \mu(h_t)$ . If the community adopts  $u$ , the policy program prescribes  $\mu(h_t, u) = \mu(h_t, \mu(h_t))$  in the next period. If the policy program is stationary, then  $\mu(h_t, \mu(h_t)) = \mu(h_t) = u$ . Thus, starting from any  $h_t$ , the continuation path,  $(u, u, u, \dots)$ , is stationary.

Henceforth, for  $\delta \in [0, 1)$ , let

$$V_\delta^* = \{w \mid \exists \text{ a DCW for which discounted payoffs are } w/(1 - \delta)\}$$

and

$$W_\delta^* = \{w \mid \exists \text{ a stationary DCW for which discounted payoffs are } w/(1 - \delta)\}$$

In other words,  $V_\delta^*$  represents the set of normalized discounted payoffs sustainable through DCWs, and  $W_\delta^*$  represents the set of normalized discounted payoffs sustainable through stationary DCWs. The normalization involves multiplying discounted payoffs by  $1 - \delta$  to keep everything on the same scale as single-period payoffs. For  $\delta = 1$ , we define  $V_\delta^*$  using limiting averages, and  $W_\delta^*$  using (constant) per-period payoffs.<sup>17</sup>

Stationary DCWs are much simpler than non-stationary DCWs, but stationarity would appear to be an extremely strong requirement (particularly inasmuch as the restriction is imposed for all histories, both on and off the “equilibrium path”). Our first result nevertheless establishes that, as long as  $U$  is compact and convex, one can restrict attention to stationary DCWs *without loss of generality* – nothing is gained by considering complex non-stationary DCWs. This observation vastly simplifies the task of finding DCWs, particularly compared with finding the subgame perfect equilibria of a dynamic voting institution.

**Theorem 1:** *If  $\delta \in [0, 1)$  and  $U$  is compact and convex, then the set of normalized payoffs associated with DCWs is the same as the set of normalized payoffs associated with stationary DCWs. That is,  $V_\delta^* = W_\delta^*$ .*

We regard Theorem 1 as somewhat surprising. By way of analogy, consider subgame perfect equilibria for stationary repeated games. It is well-known that optimal subgame perfect equilibria are not, in general, stationary (see, for example, Abreu’s [1986] discussion of “stick and carrot” punishments). This analogy is pertinent given some of the formal similarities between these concepts (for example, the proofs of many of our results use an analog of the self-generation mapping, which was developed by Abreu, Pearce, and Stachetti [1991] to analyze subgame perfect equilibria), as well as the relationship between DCWs and strategic equilibria described in Section 3.

Why then does the theorem hold? Suppose the normalized payoff vector for some non-stationary DCW is  $w$ , reflecting a current-period payoff vector of  $u' \neq w$  followed by a normalized continuation payoff vector of  $w' \in U$  (with  $w = (1 - \delta)u' + \delta w'$ ). For each  $u \in U$ , we know there is some  $w''(u) \in U$  such that  $w_i \geq (1 - \delta)u_i + \delta w''_i(u)$  for at least  $M$  agents. Since  $U$  is convex, we also know that

<sup>17</sup>By “limiting averages,” we mean that, for an infinite sequence of payoff vectors  $u^1, u^2, \dots$ , we use  $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T u^t$ . Obviously, if  $u^t = u^{t+1} = u$  for all  $t$ , then the limiting average is the per-period payoff vector  $u$ , so our treatments of  $V_\delta^*$  and  $W_\delta^*$  are mutually consistent.

$w \in U$ . Consequently, it is possible to construct a DCW that selects  $w$  in every period “on the equilibrium path,” where we follow  $u \neq w$  with the continuation payoff  $w''(u)$ , and sustain those continuation payoffs exactly as before. While this new DCW may not prescribe the same policy in every period “off the equilibrium path” (e.g.  $w''(u)$  may reflect policy choices that differ from one period to the next), precisely the same reasoning implies that one can, for any history, substitute a DCW with the same discounted payoffs and a constant policy. Intuitively, we construct a stationary DCW by making this substitution sequentially for all histories. This argument breaks down in strategic settings because, unlike here, the utility vectors achievable through deviations depend on the starting point (that is, switching from choices generating  $u'$  to choices generating  $w$  alters the subset of alternatives in  $U$  against which the current choice is tested).

Though the convexity requirement in Theorem 1 may appear somewhat demanding, one should bear in mind that it is *always* satisfied when the community is permitted to randomize over policies. In our view, there is in general no good reason to rule out randomizations; if we seek to understand why communities appear to select deterministic policies in practice, this should be a *result* and not a *restriction*. In any case, the following result shows that it is still useful to examine stationary DCWs with randomized policies even when randomization is not permitted.

**Theorem 2:** *Suppose  $X$  is convex, and  $\eta_i$  is concave for each  $i$ . Then the set of normalized payoffs associated with stationary DCWs allowing for randomization contains the set of normalized payoffs associated with DCWs (including non-stationary ones) not allowing for randomizations.*

The intuition for this result is as follows. Permitting randomizations potentially enlarges the set of sustainable outcomes by expanding both the set of feasible outcomes and the set of available punishments. It also potentially shrinks the set of sustainable outcomes by expanding the set of alternatives against which a policy program must be tested. However, under the stated conditions, no shrinkage occurs because each new alternative in this latter set (a randomization over  $X$ ) is weakly less tempting than some non-random alternative (the expected value of  $X$ ).

To illustrate the implications of these results, consider the median voter model. If the community is permitted to select randomized policies (that is, any Borel probability measure on  $X$ ), then  $U$  is compact and convex. This means Theorem 1

applies, and we can confine attention to stationary DCWs without loss of generality. If the community is not permitted to select random policies, Theorem 1 does not apply (though  $X$  is convex, strict concavity of  $\eta_i$  implies that the set  $U = \eta(X)$  is not convex). However, Theorem 2 tells us that we can still say what isn't in the set of DCW outcomes by studying stationary DCWs with randomized policies. As we'll see in Section 4.3, this observation proves useful.

## 4.2 Some sustainable outcomes

If a Condorcet winner exists in the static problem, then there is always a DCW that repeats this solution. Formally, suppose  $U$  contains a Condorcet winner  $w^C$ , and consider the policy program  $\mu(h_t) = w^C$  for all  $t, h_t$ . Then, for all  $h_t$  and  $u \in U$ ,  $C^\mu(h_t) = (w^C, w^C, \dots)R(u, w^C, w^C, \dots) = C^\mu(u, C^P(u, h_t))$ .

In a static setting, a Condorcet winner may fail to exist due to cycles in the majority preference relation. However, our motivating example (Section 2.3) suggests that it is possible to use a Condorcet cycle as the basis for constructing an equilibrium in a dynamic setting. In that example, the policies  $A, B$ , and  $C$  are sustainable because the community can use each successive element in the cycle as a contingent consequence for collective deviations from the previous element.

Our next result formalizes both of these points:

**Theorem 3:** (i) *Suppose that  $w^C$  is a Condorcet winner in  $U$ . Then, for all  $\delta \in [0, 1]$ , there exists a DCW prescribing  $w^C$  for every history.*

(ii) *Assume that  $U$  is bounded above (that is, there exists some vector  $\bar{u}$  with  $u \leq \bar{u}$  for all  $u \in U$ ). Suppose  $U$  contains a finite Condorcet cycle,  $W = \{w^1, w^2, \dots, w^K\}$  (i.e.,  $w^1 P w^2 P \dots P w^K P w^1$ ). Then there exists  $\delta^* < 1$  such that, for  $\delta \geq \delta^*$ , each element of the Condorcet cycle is the normalized payoff vector for some stationary DCW (i.e.,  $W \subseteq W_\delta^*$ ).<sup>18</sup>*

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<sup>18</sup>For  $\delta = 1$ , one can prove the following result (details available upon request). Suppose there exists some infinite sequence of payoff vectors,  $W = \{w^1, w^2, \dots\} \in U$  such that  $w^k P w^{k+1}$  for all  $k \geq 1$ . Then each element of  $W$  is the per-period payoff vector for some stationary DCW (i.e.,  $W \subseteq W_1^*$ ). This result has the following corollary: if  $U$  is strictly convex, and if each agent has strictly convex preferences, then with  $\delta = 1$  it is possible to sustain an outcome  $u \in U$  as long as there is some other  $u' \in U$  with  $u P u'$ . In other words, any payoff vector not on the lower frontier of  $U$  is sustainable. As an example, consider the policy space  $U = \{(u, u, \dots, u) | u \in [0, 1]\}$ . With  $\delta = 1$ , it is possible to sustain any outcome  $u$  in the open interval  $(0, 1]$ . The outcome  $u = 0$  is not sustainable, however.

Part (i) of Theorem 3 implies that, for the median voter model, we know there's a DCW that replicates the solution of the static choice problem, selecting  $y_M$  in every period, regardless of history. The key question is whether this is the *only* DCW outcome. We return to this question at various points below.

Part (ii) of Theorem 3 shows that, with sufficient patience (but some discounting), the community can sustain any outcome that is part of a Condorcet cycle. As an application, consider the median voter model where randomized policies are permitted. It is relatively easy to construct examples in which preferences are cyclic on the convex hull of  $\eta(X)$ . In light of Theorem 3, this means it is possible to sustain outcomes other than the median voter solution as DCWs when agents are sufficiently patient. Though cycles in  $\eta(X)$  technically involve lotteries, one can also construct non-stationary DCWs that do not involve lotteries, but that mimic lottery payoffs by selecting each policy with an appropriate frequency.

As a corollary of Theorem 3, we have the following general existence result: there is some  $\delta^* < 1$  such that a DCW exists for  $\delta > \delta^*$ . The proof of the corollary is simple. If the majority preference relation is acyclic on a compact set  $U$ , then  $U$  contains a maximal element, which is necessarily a Condorcet winner. Existence of a DCW is then assured by Theorem 3(i). Alternatively, if the majority preference relation cycles within  $U$ , Theorem 3(ii) guarantees existence of DCW for sufficiently large  $\delta$ . Existence is somewhat surprising in light of the fact that Condorcet winners often fail to exist in static settings, particularly with policy sets involving high dimensionality. After all, the dimensionality of an intertemporal policy in an infinite horizon setting is necessarily infinite. One might therefore think existence would be more difficult to guarantee. The problem is resolved by allowing for history dependence, which selectively reduces the range of potential outcomes.

Theorem 3 also provides us with a “folk theorem” for a large class of environments. Various papers (McKelvey [1976, 1979], Schofield [1978]) have shown that, under relatively weak conditions, all outcomes in the utility possibility set reside within a large Condorcet cycle. If every outcome is contained in a Condorcet cycle, then all outcomes are sustainable for sufficiently high  $\delta$ .

In light of this folk theorem, one might conclude that our solution concept has disposed of one problem (non-existence of Condorcet winners) only to replace it with another (vast multiplicity of DCWs). In response, we note the following.

First, economists are often concerned with environments in which the folk theorem does not apply, including the following. (1) In many instances, it is natural to assume that agents are impatient, in which case the set of DCW outcomes is more narrowly circumscribed (see the example in Section 5). When the appropriate rate of discounting for a particular application is unknown, one can associate each DCW with a “degree of sustainability,” based on the lowest discount factor for which it survives. This allows one to determine the features of an outcome that contribute to its sustainability, and to provide sharper characterizations of the most sustainable outcomes (see, again, the example in Section 5, where we argue that more egalitarian distributions of a fixed prize are weakly more sustainable than all other outcomes). Trivially, when a Condorcet winner exists in the static problem, this is also the most sustainable dynamic outcome. (2) In certain classes of infinitely-repeated collective choice problems, the DCW set yields sharp predictions even when the discount factor is high (see Theorem 4 below, as well as its application to the median voter model). (3) The DCW concept is also applicable to settings with overlapping generations, where a folk theorem does not emerge even if agents apply no discounting within their own lifespans. As Slavov [2006a] demonstrates, DCWs yield useful predictions in this context; indeed, the approach yields a robust explanation for the political “clout” of the elderly.

Second, the strategic approach suffers from the same multiplicity problem. There are a number of well-known folk theorems for repeated games. In political games, multiplicity can be severe even with moderate rates of discounting. For example, with repeated Downsian electoral competition, if voters’ discount factors are greater than  $1/2$ , then every sequence of policies is sustainable (see Duggan and Fey [2006]). As discussed in Section 3.2, the set of DCW outcomes is a *subset* of equilibrium outcomes for an important class of dynamic collective choice games, so multiplicity is actually *less* severe for DCWs. Moreover, just as game theorists attempt to resolve multiplicity by considering equilibrium refinements, one can similarly prune the set of DCWs (see Section 4.5 for analysis along these lines).

Notably, folk theorems are often interpreted as helpful, not problematic, in strategic settings. In particular, they tell us that, in certain circumstances, sufficiently patient agents can achieve the same outcomes with self-enforcing agreements as with enforceable contracts. Accordingly, when those agents negotiate over outcomes, they can ignore the self-enforceability requirement. A similar message

emerges here. If one views the DCW concept as a political feasibility constraint, one can interpret our folk theorem as validating (in some circumstances) the widespread practice among economists of characterizing optimal policy without regard to political considerations.<sup>19</sup>

### 4.3 Some unsustainable outcomes

The motivating example in Section 2.3 also suggests that, in certain instances, some outcomes are not sustainable. We saw in particular that, while it was possible to sustain repetitions of  $A$ ,  $B$ , or  $C$ , it was not possible to sustain repetitions of the two policies “below” the  $A$ - $B$ - $C$  cycle ( $D$  and  $E$ ). This observation suggests that DCW outcomes “unravel from the bottom”:  $E$  is not sustainable; since  $E$  is the only consequence that would deter collective deviations from  $D$ ,  $D$  is not sustainable either. Our next theorem generalizes this point.

**Theorem 4:** *Assume that  $U$  is compact and convex. Suppose there is some set  $S \subset U$  such that*

- (i)  *$R$  is transitive on  $\text{clos}(S)$ , and*
- (ii) *for any  $w' \in S$  and  $w'' \in U \setminus S$ , we have  $w'' P w'$ .*

*Then, for  $\delta \in [0, 1)$ , there does not exist a DCW sustaining any outcome in  $S$  (that is,  $S \cap V_\delta^* = \emptyset$ ).*

The proof of Theorem 4 is more subtle than one might expect. Based on our motivating example, one might be inclined to argue that, if  $w$  is sustainable, then the future consequence of a collective deviation from  $w$  must be another policy  $w'$  with  $w P w'$ ; likewise, if  $w'$  is sustainable, then the future consequence of a collective deviation from  $w'$  must be yet another policy  $w''$  with  $w' P w''$ , and so forth. Thus, if  $w \in S$ , one eventually runs out of potential consequences. This simple intuition is, however, incomplete, in that the consequence that sustains  $w$  could in principle be another policy  $w'$  with  $w' P w$ .<sup>20</sup> Theorem 4 imposes additional structure that rules out this possibility in the pertinent instances. To see why, suppose we wish

<sup>19</sup>As we demonstrate in the Appendix, if  $U$  is compact, the set of sustainable DCW payoffs is also compact. Thus, as long as the social welfare function is continuous, a constrained optimum exists because the set of politically feasible outcomes is compact.

<sup>20</sup>For example, suppose  $w = (10, 10, 10)$ ,  $u = (20, 11, 0)$ ,  $w' = (20, 7, 12)$ , and  $\delta = \frac{1}{2}$ . Then plainly  $w' P w$ , but  $w P ((1 - \delta)u + \delta w') = (20, 9, 6)$ .

to sustain some  $w \in S$ . Let's try to construct a DCW that punishes a deviation to some  $u \in U \setminus S$  with a punishment  $w'$  that majority defeats  $w$ . Let  $v = (1 - \delta)u + \delta w'$  (this is the normalized payoff vector when  $u$  is chosen). Obviously,  $w$  must majority defeat  $v$ , which means  $v \in S$  by condition (ii). We can't have  $v \in S$  when  $w' \in U \setminus S$  (because  $v$  always majority defeats either  $u$  or  $w'$ , both of which are in  $U \setminus S$ ). But if  $w' \in S$ , then  $v$  majority defeats  $w'$  (since  $u$  majority defeats  $w'$ ), which means  $v$  majority defeats  $w$  by transitivity.

Theorem 4 has an immediate corollary: if  $R$  is transitive on  $U$ , if  $U$  is compact and convex, and if  $U$  contains a static Condorcet winner,  $w^C$ , that is *strictly* majority preferred to all other alternatives, then  $w^C$  is the *only* normalized payoff vector associated with a DCW.<sup>21</sup> To illustrate, consider again the policy space  $U = \{(u, u, \dots, u) \mid u \in [0, 1]\}$ . With no discounting, we know that any  $u > 0$  is sustainable in a DCW (footnote 10). However, with any degree of discounting, Theorem 4 implies that the only possible outcome is  $u = 1$ .

For an application of greater economic interest, we return to the median voter model, but specialize to the case where  $\eta_i(x) = -\beta(y_i - x)^2$  (which implies no second-order heterogeneity). Suppose that randomized policies are allowed. Banks and Duggan [2006] show that  $R$  coincides exactly with the preference relation of the median voter (agent  $M$ ),  $R_M$ , over  $U$ . Accordingly, Theorem 4 tells us that, for this special case, the static solution,  $\eta(y_M)$ , is the *only* normalized payoff vector associated with a DCW. The same conclusion holds when randomization is *not* permitted: from Theorems 3 and 4, we know that  $\eta(y_M)$  is the only normalized payoff vector associated with a stationary DCW when randomization is permitted; from Theorem 2 we know that no other normalized payoffs are sustainable when randomization is disallowed; and from Theorem 3 we know that  $\eta(y_M)$  remains sustainable when randomization is disallowed.

In both of the preceding examples, the unique DCW outcome coincides with the static Condorcet winner. In fact, whenever the DCW outcome is unique, it must be a static Condorcet winner.<sup>22</sup> Thus, existence of a static Condorcet winner is a necessary condition for uniqueness of the DCW outcome.

<sup>21</sup>To see why, let  $S = U \setminus \{w^C\}$ , and apply the theorem.

<sup>22</sup>Uniqueness implies that the continuation utility vector is independent of the current outcome. But then, for any DCW  $\mu$ , the choice prescribed for the current period must majority-defeat all other alternatives, considering only current payoffs.

Theorem 3 and its corollary show that the existence of a Condorcet cycle helps to assure the existence DCWs if individuals are sufficiently patient. Conversely, when the one-shot payoff space does not contain a Condorcet winner, non-existence is guaranteed when agents are sufficiently impatient.

**Theorem 5:** *Suppose that  $U$  is compact. If  $U$  does not contain a Condorcet winner, then there is some  $\delta^0 > 0$  such that DCWs do not exist for  $\delta < \delta^0$ .*

This result is intuitive. With no Condorcet winner, for every element of the policy set there is another element that *strictly* defeats it. With  $\delta$  near zero, members of the group place almost all weight on the first period, so the outcome is governed by the one-shot comparison.

#### 4.4 Discounting

The general existence result that arises as a corollary of Theorem 3, and the nonexistence result in Theorem 5, underscore the importance of the discount factor for collective choice. It is natural to wonder how the set of sustainable outcomes changes with this parameter. As our next result shows, when  $U$  is convex, the set of sustainable normalized discounted payoffs expands monotonically as agents become more patient.

**Theorem 6:** *Assume that  $U$  is compact and convex. Consider two discount factors,  $\delta$  and  $\delta'$ , with  $1 > \delta' > \delta$ . Then the set of normalized payoffs sustainable through DCWs is weakly larger with  $\delta'$  than with  $\delta$  (that is,  $V_{\delta'}^* \subseteq V_{\delta}^*$ ).*

**Remark:** Since  $U$  is convex, the same statement holds for stationary DCWs by Theorem 1.

When  $U$  is compact, convex, and contains no Condorcet winner, Theorems 3 (the general existence corollary), 5, and 6 together imply that there is a single threshold ( $\delta^* = \delta^0 \in (0, 1)$ ) above which DCWs exist, and below which they do not. There is no reason to think that the threshold is particularly high or low; indeed, for the pure distributional problem considered in Section 5, it is  $\frac{1}{2}$ .

How should one interpret non-existence? Recall that our objective in formulating the DCW concept is to identify outcomes that are arguably robust across a range of majoritarian institutions. The absence of robust outcomes implies that collective

choices must hinge on the specific features of those institutions. While our approach does not completely resolve the problem of non-existence for Condorcet winners, it considerably expands the range of environments for which the non-game-theoretic identification of institutionally robust outcomes is a viable alternative to the game theoretic, and necessarily institution-specific, mode of analysis.

#### 4.5 Efficiency and renegotiation-proofness

In infinitely-repeated games, it is often difficult to sustain the most efficient outcomes in the utility possibility set unless the discount factor is sufficiently high. By analogy with subgame perfection, one might expect to have some difficulty sustaining efficient outcomes as DCWs as well. However, it turns out that the set of DCWs has nice efficiency properties. In particular, outcomes that are efficient within the set of DCWs are also efficient within the utility possibility set.

For any compact set  $W \subseteq U$ , let  $F(W)$  denote the *weakly efficient frontier*, defined as follows:

$$F(W) = \{w \in W \mid \text{for any other } w' \in W, w_i \geq w'_i \text{ for some } i\}$$

Similarly, let  $F^s(W)$  denote the *strictly efficient frontier*, defined as follows:

$$F^s(W) = \{w \in W \mid \text{for any other } w' \in W, w_i > w'_i \text{ for some } i\}$$

**Theorem 7:** *Assume  $U$  is compact and convex. The weakly efficient frontier of the set of normalized payoffs associated with DCWs is weakly efficient within  $U$ , and the strictly efficient frontier of the set of normalized payoffs associated with DCWs is strictly efficient within  $U$  (that is,  $F(V_\delta^*) \subseteq F(U)$ , and  $F^s(V_\delta^*) \subseteq F^s(U)$ ).*

**Remark:** As is clear from the proof of the theorem, the stated properties hold for *stationary* DCWs regardless of whether  $U$  is convex.

Theorem 7 also motivates consideration of refinements. In the context of Nash equilibria, one common refinement is to focus on the efficient frontier of the equilibrium set (the Pareto refinement). Theorem 7 tells us that, in the context of DCWs, the Pareto refinement always places the community on the efficient frontier of  $U$ . This is a natural refinement inasmuch as no member of the community would oppose efforts to coordinate on a Pareto improvement.

There is, however, a well-known problem with the Pareto refinement: subgame perfect equilibria that generate efficient payoffs on the equilibrium path often entail highly inefficient outcomes (used as punishments) off the equilibrium path. These inefficient outcomes are frequently viewed as vulnerable to renegotiation. The literature has explored various notions of renegotiation-proof equilibria for dynamic games, beginning with Bernheim and Ray [1989] and Farrell and Maskin [1989]. The definition of a renegotiation-proof equilibrium is relatively uncontroversial for finite horizon games. Formulating a concept for the infinite horizon case is more controversial. One particularly strong notion of renegotiation-proofness requires outcomes to remain on the efficient frontier of the feasible payoff set, both on and off the equilibrium path. With such a strong requirement, existence is not generally guaranteed for infinitely repeated games. For equilibria that correspond to DCWs, however, it turns out that this requirement is quite easy to satisfy.

To develop this line of analysis, we need some additional definitions. When  $U$  is compact, each agent's payoff is bounded below. Let  $\underline{u}_i$  denote the lowest feasible payoff for agent  $i$ .

**Definition:** The set  $U$  is characterized by *free disposal* if, for all  $u \in U$  and  $u' \neq u$  such that  $\underline{u} \leq u' \leq u$ , we have  $u' \in U$ . The set  $U$  is characterized by *strict comprehensiveness* if, in addition, there exists  $u'' > u'$  such that  $u'' \in U$ .

Both definitions are standard. Free disposal implies that it is always possible to throw away payoffs, subject to the lower bound. Strict comprehensiveness implies that reductions in one agent's payoff always permit increases in others' payoffs. Notice that strict comprehensiveness implies free disposal.

**Definition:** A DCW  $\mu$  satisfies *universal weak efficiency (UWE)* if, for all  $h_t$ , the normalized continuation payoff lies on the weakly efficient frontier of  $\text{Co}(U)$ .<sup>23</sup> A DCW  $\mu$  satisfies *universal strict efficiency (USE)* if, for all  $h_t$ , the normalized continuation payoff vector lies on the strictly efficient frontier of  $\text{Co}(U)$ .

Let  $E_\delta$  denote the set of normalized payoff vectors sustainable in DCWs satisfying universal weak efficiency. Likewise, let  $E_\delta^s$  denote the set of normalized payoff vectors sustainable in DCWs satisfying universal strict efficiency. Note that any strategic equilibrium corresponding to a universally efficient DCW (weak or strict)

<sup>23</sup>For any set  $A$ ,  $\text{Co}(A)$  denotes the convex hull of  $A$ .

is renegotiation-proof in the very strong sense that the community always remains on the efficient frontier, irrespective of history. To state our next result, we also define  $W_\delta^*(A)$  as the set of normalized payoffs corresponding to stationary DCWs when the feasible payoff set is  $A$ . Clearly,  $W_\delta^*(U)$  and  $W_\delta^*$  (as defined in Section 4) are identical; we have simply suppressed the dependence on  $U$  until this point to simplify notation.

**Theorem 8:** *Assume  $U$  is compact and convex.*

- (i) *If  $U$  is characterized by free disposal, then the following sets of normalized payoff vectors are equivalent: those associated with DCWs satisfying universal weak efficiency, those associated with weakly efficient DCWs, and those associated with stationary DCWs when choices are restricted to the weakly efficient frontier of  $U$  (in other words,  $E_\delta = F(V_\delta^*) = W_\delta^*(F(U))$ ).*
- (ii) *If  $U$  is characterized by strict comprehensiveness, then the following sets of normalized payoff vectors are equivalent: those associated with DCWs satisfying universal strict efficiency, those associated with strictly efficient DCWs, and those associated with stationary DCWs when choices are restricted to the strictly efficient frontier of  $U$  (in other words,  $E_\delta^s = F^s(V_\delta^*) = W_\delta^*(F^s(U))$ ).*

Theorem 8 has two implications. First, the set of outcomes found in universally efficient DCWs is exactly the same as the efficient frontier of the DCW set. In other words, any efficient DCW outcome can be sustained as a universally efficient DCW. The intuition for this result is simple. To discourage a collective deviation, a consequence need only make a majority of individuals worse off. If a consequence is inefficient, one can replace it with another consequence that gives the same payoffs to members of the decisive majority, and higher payoffs to individuals who are not members of this majority. By the argument used to prove Theorem 7, this alternative consequence is also sustainable as a DCW. Therefore, one can continue to increase the payoff to individuals who are not members of the decisive majority until the punishment lies on the efficient frontier.

Second, Theorem 8 implies that the set of universally efficient DCWs is exactly equal to the set of DCWs obtained by first throwing away all inefficient feasible outcomes, and then finding the stationary DCW set. To illustrate this result, imagine the community faces the problem of dividing a dollar among its members,

and that it is free to throw away any portion of the dollar. Any outcome in which the entire dollar is divided among agents is on the efficient frontier, and any outcome in which some of the dollar is thrown away is inefficient. Theorem 8 tells us that to find the universally efficient DCW set for this problem, we can ignore all the inefficient outcomes (in which a portion of the dollar is destroyed) and simply solve for DCWs under the assumption that the policy space is just the unit simplex. Note that this result holds even when the efficient frontier of the utility possibility set is not convex (provided the underlying set  $U$  is convex).

As an application, we return again to the median voter model. So far, we've shown that the static Condorcet winner,  $\eta(y_M)$ , is always sustainable through a DCW, and that it's the only sustainable outcome when preferences exhibit no second-order heterogeneity. However, we also know that, more generally, there are other sustainable outcomes. Our next result shows that, with modest assumptions concerning preferences, universal weak efficiency rules out these other outcomes, leaving only the repeated static solution.

**Theorem 9:** *Consider the median voter model. Assume preferences satisfy the weak single-crossing property (specifically, if  $\eta_i(y') \geq \eta_i(y'')$  for  $y' < y''$ , then  $\eta_j(y') \geq \eta_j(y'')$  for  $j < i$ ). Assume also that  $\delta \in (0, 1)$ . The static Condorcet winner is the only normalized payoff vector sustainable in a DCW satisfying universal weak efficiency (that is,  $E_\delta = \{\eta(y_M)\}$ ).*

## 5 Application: Pure Distributive Politics

As an application, we now consider the classical problem of dividing a fixed payoff (henceforth a “dollar”) among a number of agents (e.g., Baron and Ferejohn [1987,1989], Epple and Riordan [1987]). The problem is important because it involves pure distributive politics. It is subtle because there is no Condorcet winner; for any division of the dollar, there exists another division that makes a majority strictly better off.

Since every possible choice belongs to a Condorcet cycle, this is an instance where our folk theorem (Theorem 3) applies. Our objective here is to characterize the DCW set as a function of the discount factor. We offer a complete characterization for the case of  $N = 3$  and a partial characterization for the case of  $N > 3$ , and we identify implications for distributional policy.

Formally, we assume that each individual's utility is equal to the share of the dollar received. Thus, the set of feasible payoffs is the unit simplex:

$$U = \Delta^N \equiv \left\{ u \in \mathbb{R}_+^N \mid \sum_{i=1}^N u_i = 1 \right\}.$$

Note that we do not allow the community to throw away any fraction of the dollar. However, as discussed in the previous section, this is equivalent to allowing for free disposal provided that we restrict attention to universally efficient DCWs. Define

$$\Delta_\delta^N = \left\{ u \in \Delta^N \mid \forall A \subset \{1, \dots, N\} \text{ with } |A| = M, \sum_{i \in A} u_i \geq 1 - \delta \right\}$$

**Theorem 10:** (i) *The set of normalized payoffs associated with DCWs is contained in  $\Delta_\delta^N$  (that is,  $V_\delta^* \subseteq \Delta_\delta^N$ ).*

(ii) *If  $\delta < \frac{1}{2}$ , no DCWs exist.*

(iii) *If  $\delta \geq \frac{N-1}{N+1}$ , the set of normalized payoffs associated with DCWs is exactly  $\Delta_\delta^N$  (that is,  $V_\delta^* = \Delta_\delta^N$ ).*

Parts (i) and (iii) of Theorem 10 imply that if  $N = 3$ , all policies giving at least  $(1 - \delta)$  to every majority are in  $W_\delta^*$  provided  $\delta \geq \frac{1}{2}$  (if  $N = 3$ ,  $\frac{N-1}{N+1} = \frac{1}{2}$ ). We illustrate these constraints in Figure 1 for various values of  $\delta$ . The innermost set shows the divisions that are sustainable when  $\delta = .5$ . This is contained in the set that is sustainable for  $\delta = .8$ , which in turn is contained in the set that is sustainable for  $\delta = 1$ . As discussed in Section 4.2, one can associate each DCW with a “degree of sustainability,” based on the lowest discount factor for which it survives (with survival at lower discount factors implying greater sustainability). Here, the most sustainable outcomes are those in the innermost set.

Note that, for  $\delta \in [\frac{1}{2}, 1)$ , some outcomes are sustainable, and some are not. This illustrates as point made earlier: our solution concept is more discerning than, say, repeated Downsian competition, which can produce any outcome as long as  $\delta > \frac{1}{2}$  (see Duggan and Fey [2006], who also refine the equilibrium set).

If we interpret each individual as representing a social class (e.g. poor, middle class, and wealthy), several interesting and significant conclusions follow from an inspection of Figure 1. (1) No alternative is more sustainable than the egalitarian outcome ( $u^*$ ). (2) Starting from any other alternative, movements toward egalitarianism always (weakly) improve sustainability. We illustrate this in the figure

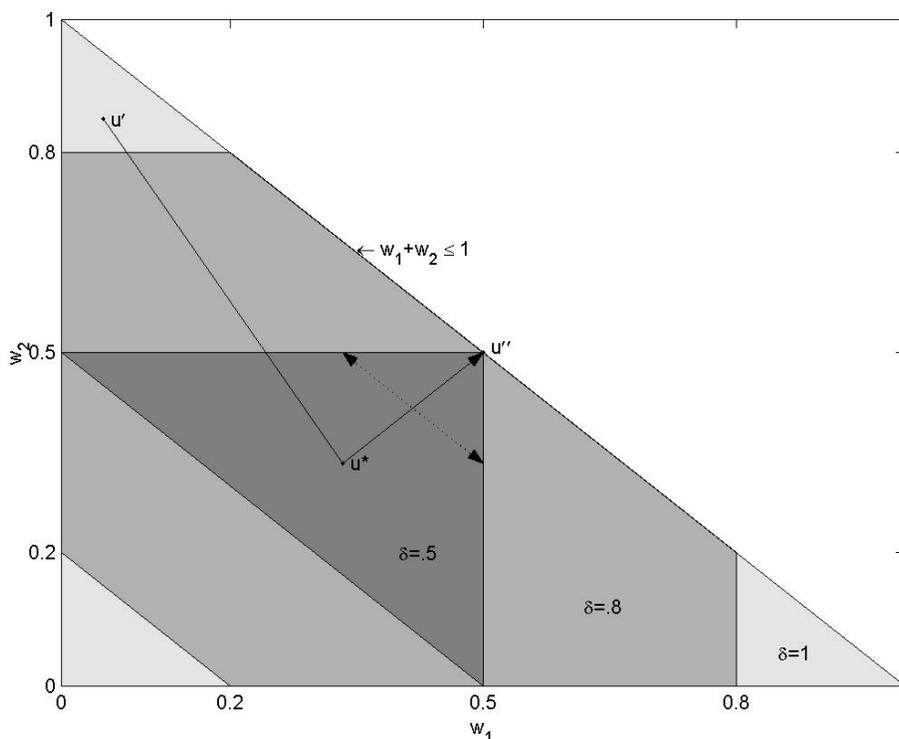


Figure 1: Division of a Fixed Payoff

by drawing a straight line from an arbitrary point ( $u'$ ) to the egalitarian outcome ( $u^*$ ). As we move along the line toward  $u^*$ , we cross into payoff sets sustained at progressively lower discount factors. These first two conclusions suggest that political considerations favor equality. There are, of course, unequal outcomes that are as sustainable as perfect egalitarianism (e.g., point  $u''$ ). However: (3) penalizing one party requires a measure of equality between the other two. Note, for example, that as one moves from  $u^*$  to  $u''$ , decreasing the payoff to agent 3, the scope for inequality between agents 1 and 2 (as indicated by the double-headed arrow) within the set of most sustainable outcomes declines monotonically. This leads to our next conclusion: (4) it is easier to sustain outcomes with extremely poor minorities (who receive essentially nothing) than ones with extremely wealthy minorities. That is, political feasibility places a limit on wealth, but not on poverty. To see this point, note that, for any  $\delta \geq \frac{1}{2}$ , no individual receives more than  $\delta$ , but there are DCWs

in which one individual is arbitrarily poor. Within the most sustainable set, the Gini coefficient lies in the interval  $[0, \frac{1}{2}]$ ; the highest degree of inequality is attained for outcomes in which one party receives nothing, and the other two split the prize equally.

Finding DCW outcomes when  $N > 3$  and  $\frac{1}{2} \leq \delta \leq \frac{N-1}{N+1}$  is more complicated. By theorem 10(i), we know that  $W_\delta^* \subseteq \Delta_\delta^N$  for  $\delta$  in this range. However, by way of example, one can show that not all policies in  $\Delta_\delta^N$  are sustainable.

## 6 Concluding remarks

The value of any solution concept lies in the insights it supplies when applied to concrete economic problems. Here we have explored two standard applications: the “median voter” problem, and division of a fixed surplus. For the first application, we’ve identified both a preference restriction and an equilibrium refinement that justify focusing on the static solution even when the setting is dynamic. For the second application, we’ve shown that political considerations (weakly) favor equality, and that political feasibility places upper limits on wealth, but does not place lower limits on poverty. For more elaborate applications to other significant economic issues, see Bernheim and Slavov [2004] (who study the selection of incentive compatible tax systems), Slavov [2006a] (who studies the political power of the elderly), and Slavov [2006b] (who examines private and public provision of public goods).

Although we have defined DCWs for a class of simple repeated collective choice problems with infinitely-lived agents, our definition extends in a straightforward manner to other dynamic settings, including those with overlapping generations (see Slavov [2006a]) and/or state variables. Similarly, the concept is easily generalized to encompass environments with publicly observed, payoff-irrelevant random events. This generalization permits one to convexify the continuation payoff set  $W_\delta^*$  through public randomization, thereby facilitating numerical computation of DCW sets (see Slavov [2006b]).

Other extensions are also possible. In defining our solution concept, we have used the majority preference relation. One could define similar concepts for other relations (e.g. a supermajority requirement). A careful reading of our proofs reveals that some results rely only on completeness and continuity of the majority preference relation, while others also require the Pareto property. As one of our referees noted, one could alternatively reinterpret our framework as describing the

choices of a single dynamically inconsistent decision-maker whose preferences are also intransitive.

## Appendix

To prove several of our theorems, we use an analog of the *self-generation map*, which is a tool used to simplify the analysis of subgame-perfect equilibrium (see Abreu, Pearce, and Stacchetti [1990]). We define the self-generation map for DCWs as follows: for any set  $W \subseteq \text{Co}(U)$  (where  $\text{Co}(U)$  denotes the convex hull of  $U$ ),

$$\Psi_\delta(W) = \{w \mid \text{there exists } u'' \in U \text{ and } w'' \in W \text{ such that } w = (1 - \delta)u'' + \delta w''$$

$$\text{and, } \forall u \in U, \text{ there exists } w' \in W \text{ with } wR((1 - \delta)u + \delta w')\}$$

In words, the self-generation map identifies the set of normalized payoffs that are achievable when normalized continuation payoffs must lie in the set  $W$ . A normalized payoff vector  $w$  can be achieved provided that (i)  $w$  can be decomposed into some feasible current outcome  $u''$  and some  $w''$  in the feasible normalized continuation set  $W$ , and (ii) for any other current outcome  $u$ , there is another feasible normalized continuation payoff vector  $w'$  such that a majority prefers  $u''$  followed by  $w''$  to  $u$  followed by  $w'$ . In a stationary environment, the set of feasible normalized continuation payoffs must be the same as the set of achievable normalized payoffs. Consequently, the set of normalized equilibrium payoffs is a fixed point of the self-generation map.

For a stationary DCW, the prescribed current payoff and the normalized continuation payoff are always the same. Accordingly, we define the stationary self-generation map as follows: for any set  $W \subseteq U$ ,

$$\Phi_\delta(W) = \{w \in W \mid \forall u \in U, \text{ there exists } w' \in W \text{ with } wR((1 - \delta)u + \delta w')\}$$

We demonstrate below in Lemma 5 that  $V_\delta^*$  and  $W_\delta^*$  correspond to the largest fixed points of the self-generation maps  $\Psi_\delta$  and  $\Phi_\delta$ , respectively. To obtain the largest fixed point of  $\Psi_\delta$ , one natural strategy is to start with the set of feasible normalized payoff vectors,  $\Omega_\delta$ , and iteratively apply  $\Psi_\delta$ , at each step eliminating infeasible continuation payoffs until convergence is achieved. Formally,

$$\Omega_\delta \equiv \left\{ w \mid \exists (u^1, u^2, \dots) \text{ with } u^t \in U \forall t \text{ such that } w = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u^t \right\}$$

We note that  $\Omega_\delta = U$  when  $U$  is convex. Define  $\Psi_\delta^1 = \Psi_\delta(\Omega_\delta)$ , and recursively  $\Psi_\delta^t = \Psi_\delta(\Psi_\delta^{t-1})$ . Let

$$\Psi_\delta^\infty = \cap_{t=0}^{\infty} \Psi_\delta^t$$

Likewise, to obtain the largest fixed point of  $\Phi_\delta$ , one might start with the payoff set  $U$  and iteratively apply  $\Phi_\delta$  until convergence is achieved. Define  $\Phi_\delta^1 = \Phi_\delta(U)$ , and recursively  $\Phi_\delta^t = \Phi_\delta(\Phi_\delta^{t-1})$ . Let

$$\Phi_\delta^\infty = \bigcap_{t=0}^\infty \Phi_\delta^t$$

Our analysis begins with a series of lemmas.

**Lemma 1:** *Consider a sequence  $u^1, u^2, \dots \in \mathbb{R}^N$  converging to  $u^*$ . Consider a second sequence  $w^1, w^2, \dots \in \mathbb{R}^N$  with at least one limit point, and  $u^n R w^n$  for each  $n$ . Then, for any limit point  $w^*$  of the second sequence, we have  $u^* R w^*$ .*

**Proof:** The lemma reflects a well-known property (continuity of the majority preference relation), but is also easy to establish directly. Since  $N$  is finite, we can select some subsequence indexed by  $n_t$  with  $w^{n_t} \rightarrow w^*$  such that the set  $\{i \mid u_i^{n_t} \geq w_i^{n_t}\} \equiv I^*$  is the same for all  $t$ . We know that  $\#I^* \geq M$ . Taking limits, we have  $u_i^* \geq w_i^*$  for  $i \in I^*$ . ■

**Lemma 2:** (a) *If  $A \subseteq B \subseteq \Omega_\delta$ , then  $\Psi_\delta(A) \subseteq \Psi_\delta(B)$ .* (b) *If  $A \subseteq B \subseteq U$ , then  $\Phi_\delta(A) \subseteq \Phi_\delta(B)$ .* (c) *If  $A \subseteq \Omega_\delta$  and  $A \subseteq \Psi_\delta(A)$ , then  $A \subseteq \Psi_\delta^\infty$ .* (d) *If  $A \subseteq U$  and  $A \subseteq \Phi_\delta(A)$ , then  $A \subseteq \Phi_\delta^\infty$ .* (e)  $\Psi_\delta^{t+1} \subseteq \Psi_\delta^t \subseteq U$  for all  $t \geq 1$ . (f)  $\Phi_\delta^{t+1} \subseteq \Phi_\delta^t \subseteq U$  for all  $t \geq 1$ .

**Proof:** (a) Consider any  $w \in \Psi_\delta(A)$ . We know there exists  $u'' \in U$  and  $w'' \in A$  with  $w = (1 - \delta)u'' + \delta w''$ ; obviously,  $w'' \in B$ . Moreover, for each  $u \in U$ , there exists  $w' \in A$  such that  $w R ((1 - \delta)u + \delta w')$ . Since  $A \subseteq B$ , we have  $w' \in B$ . Since this holds for all  $u \in U$ , we have  $w \in \Psi_\delta(B)$ .

(b) Consider any  $w \in \Phi_\delta(A)$ . For each  $u \in U$ , there exists  $w' \in A$  such that  $w R ((1 - \delta)u + \delta w')$ . Since  $A \subseteq B$ , we have  $w' \in B$ . Since this holds for all  $u \in U$ , we have  $w \in \Phi_\delta(B)$ .

(c) By part (a) (taking  $B = \Omega_\delta$ ),  $A \subseteq \Psi_\delta(A) \subseteq \Psi_\delta(\Omega_\delta) = \Psi_\delta^1$ . Now assume that  $A \subseteq \Psi_\delta^{t-1}$ . Then, by part (a) (taking  $B = \Psi_\delta^{t-1}$ ),  $A \subseteq \Psi_\delta(A) \subseteq \Psi_\delta(\Psi_\delta^{t-1}) = \Psi_\delta^t$ . Thus, by induction,  $A \subseteq \Psi_\delta^t$  for all  $t$ , which implies  $A \subseteq \Psi_\delta^\infty$ .

(d) Same as (c), substituting  $U$  for  $\Omega_\delta$  and  $\Phi$  for  $\Psi$ .

(e) By definition,  $\Psi_\delta^1 = \Psi_\delta(\Omega_\delta) \subseteq \Omega_\delta$ . Applying part (a) (taking  $A = \Psi_\delta^1$  and  $B = \Omega_\delta$ ), we have  $\Psi_\delta^2 \subseteq \Psi_\delta^1$ . Now apply induction: assuming  $\Psi_\delta^t \subseteq \Psi_\delta^{t-1}$  and applying part (a) (taking  $A = \Psi_\delta^t$  and  $B = \Psi_\delta^{t-1}$ ), we have  $\Psi_\delta^{t+1} \subseteq \Psi_\delta^t$ .

(f) Follows immediately from the fact that, by construction,  $\Phi_\delta(W) \subseteq W$  for any set  $W$ . ■

**Lemma 3:** *Assume  $U$  is compact. (a) Consider  $W \subseteq \Omega_\delta$ ; if  $W$  is compact, so is  $\Psi_\delta(W)$ . (b) Consider  $W \subseteq U$ ; if  $W$  is compact, so is  $\Phi_\delta(W)$ . (c)  $\Psi_\delta^t$  is compact for all  $t$ , including  $t = \infty$ . (d)  $\Phi_\delta^t$  is compact for all  $t$ , including  $t = \infty$ .*

**Proof:** (a) Consider any convergent sequence  $w_1, w_2, \dots \in \Psi_\delta(W)$  with limit point  $w_\infty$ . For each  $w_t$ , there exists  $u_t'' \in U$  and  $w_t'' \in W$  such that  $w_t = (1 - \delta)u_t'' + \delta w_t''$ . Since  $U$  is compact, it contains all limit points of  $u_t''$ ; likewise, since  $W$  is compact, it contains all limit points of  $w_t''$ . Consequently, there exists  $u_\infty'' \in U$  and  $w_\infty'' \in W$  such that  $w_\infty = (1 - \delta)u_\infty'' + \delta w_\infty''$ .

For each  $w_t$ , there exists a function  $w_t' : U \rightarrow W$  such that,  $\forall u \in U$ , we have  $w_t R((1 - \delta)u + \delta w_t'(u))$ . For each  $u$ , let  $w_\infty'(u)$  be a limit point of the sequence  $w_t'(u)$ . Since  $W$  is compact,  $w_\infty'(u) \in W$ . Moreover,  $\forall u \in U$ , we have  $w_\infty R((1 - \delta)u + \delta w_\infty'(u))$  (Lemma 1). This establishes that  $\Psi_\delta(W)$  is closed (it contains  $w_\infty$ ).

It is easy to verify that, since  $U$  is compact,  $\Omega_\delta$  is compact. For  $W \subseteq \Omega_\delta$ , we have  $\Psi_\delta(W) \subseteq \Omega_\delta$  by construction. Since  $\Psi_\delta(W)$  is a closed subset of a compact set, it is compact.

(b) We establish that  $\Phi_\delta(W)$  is closed using the same argument as in the second paragraph of the proof of part (a). Since  $\Phi_\delta(W) \subseteq W$  by construction, and since  $W$  is compact by assumption,  $\Phi_\delta(W)$  is compact.

(c) By induction, part (a) (along with the compactness of  $\Omega_\delta$ ) implies that  $\Psi_\delta^t$  is compact for all  $t$ . Part (e) of Lemma 2 implies that the sequence  $\Psi_\delta^1, \Psi_\delta^2, \dots$  is nested. Since  $\Psi_\delta^\infty$  is the infinite intersection of nested compact sets, it is compact.

(d) Same as (c), substituting  $\Phi$  for  $\Psi$ ,  $U$  for  $\Omega_\delta$ , part (b) for part (a), and part (f) of Lemma 2 for part (e). ■

**Lemma 4:** *If  $U$  is compact, then (a)  $\Psi_\delta(\Psi_\delta^\infty) = \Psi_\delta^\infty$  and (b)  $\Phi_\delta(\Phi_\delta^\infty) = \Phi_\delta^\infty$ .*

**Proof:** (a) First we argue that  $\Psi_\delta(\Psi_\delta^\infty) \subseteq \Psi_\delta^\infty$ . From part (a) of Lemma 2 and the fact that  $\Psi_\delta^\infty \subseteq \Psi_\delta^t$  for all  $t$ , we have  $\Psi_\delta(\Psi_\delta^\infty) \subseteq \Psi_\delta^{t+1}$  for all  $t$ , from which the conclusion follows directly.

Next we argue that  $\Psi_\delta(\Psi_\delta^\infty) \supseteq \Psi_\delta^\infty$ . Assume that  $\Psi_\delta^\infty$  is non-empty; otherwise the statement is trivial. Take any  $w \in \Psi_\delta^\infty$ . We show that  $w \in \Psi_\delta(\Psi_\delta^\infty)$ , in three steps.

Step 1: We claim that, for any sequence  $w_1, w_2, \dots$  converging to some limit  $w_\infty$  with  $w_{t+1} \in \Psi_\delta^t$  for each  $t$ , we have  $w_\infty \in \Psi_\delta^\infty$ . Since each  $\Psi_\delta^t$  contains  $w_{t+1}$ , and since  $\Psi_\delta^\tau \subseteq \Psi_\delta^t$  for  $\tau > t$  (Lemma 2 part (e)), we have  $w_\tau \in \Psi_\delta^t$  for all  $\tau \geq t+1$ ; since  $\Psi_\delta^t$  is also compact, it therefore contains  $w_\infty$ . Since this holds for all  $t$ ,  $w_\infty \in \Psi_\delta^\infty$ .

Step 2: We claim that there exists  $u''_\infty \in U$  and  $w''_\infty \in \Psi_\delta^\infty$  with  $w = (1 - \delta)u''_\infty + \delta w''_\infty$ . For all  $t \geq 1$ , we know  $w \in \Psi_\delta^t$ , so for each  $t$  there exists  $u''_t \in U$  and  $w''_t \in \Psi_\delta^{t-1}$  with  $w = (1 - \delta)u''_t + \delta w''_t$ . Consider any subsequence for which  $u''_t$  and  $w''_t$  converge to limits  $u''_\infty$  and  $w''_\infty$  (existence is assured since  $\Omega_\delta$  is compact). Clearly,  $u''_\infty \in U$  and (from step 1)  $w''_\infty \in \Psi_\delta^\infty$ .

Step 3: We claim that, for each  $u \in U$ , there exists  $w'_\infty(u) \in \Psi_\delta^\infty$  such that  $wR((1 - \delta)u + \delta w'_\infty(u))$ . For each  $t \geq 1$ , we know  $w \in \Psi_\delta^t$ , so for each  $t$  there exists a function  $w'_t : U \rightarrow \Psi_\delta^{t-1}$  such that for all  $u \in U$ , we have  $wR((1 - \delta)u + \delta w'_t(u))$ . For each  $u$ , let  $w'_\infty(u)$  be a limit point of the subsequence  $w'_t(u)$ . By step 1,  $w'_\infty(u) \in \Psi_\delta^\infty$ . By Lemma 1, we have  $wR((1 - \delta)u + \delta w'_\infty(u))$ .

Together, steps 2 and 3 imply  $w \in \Psi_\delta(\Psi_\delta^\infty)$ , as required.

(b) To prove  $\Phi_\delta(\Phi_\delta^\infty) \subseteq \Phi_\delta^\infty$ , we argue exactly as in part (a), substituting  $\Phi$  for  $\Psi$ , and Lemma 2, part (b) for Lemma 2, part (a). To prove  $\Phi_\delta(\Phi_\delta^\infty) \supseteq \Phi_\delta^\infty$ , we argue as in steps 1 and 3 of part (a), substituting  $\Phi$  for  $\Psi$ , and Lemma 2, part (f) for Lemma 2, part (e). ■

Henceforth, let  $D = (1, \delta, \delta^2, \dots)'$ . For any feasible policy sequence  $\gamma = (u^0, u^1, \dots)$ , the normalized payoff vector is  $(1 - \delta)\gamma D = (1 - \delta) \sum_{t=0}^\infty \delta^t u^t$ .

**Lemma 5:** (a) If  $W \subseteq \Psi_\delta(W)$  for some  $W \subseteq \Omega_\delta$ , then  $W \subseteq V_\delta^*$ . (b) If  $W \subseteq \Phi_\delta(W)$  for some  $W \subseteq U$ , then  $W \subseteq W_\delta^*$ .

**Proof:** (a) Consider any  $w \in W$ . We will construct functions  $\mu : \cup_{\tau=0}^\infty H_\tau \rightarrow U$  and  $\hat{w} : \cup_{\tau=0}^\infty H_\tau \rightarrow W$  such that  $\hat{w}(h_0) = w$ , and such that for all  $\tau \geq 0$  and  $h_\tau \in H_\tau$  we have (i)  $\hat{w}(h_\tau) = (1 - \delta)\mu(h_\tau) + \delta \hat{w}(h_\tau, \mu(h_\tau))$ , and (ii)  $\hat{w}(h_\tau)R((1 - \delta)u + \delta \hat{w}(h_\tau, u))$  for all  $u \in U$ .

Since  $w \in \Psi_\delta(W)$ , we know that  $\exists u'' \in U$  and  $w'' \in W$  such that  $w = (1 - \delta)u'' + \delta w''$  and,  $\forall u \in U$ , there exists  $w'(u) \in W$  with  $wR((1 - \delta)u + \delta w'(u))$ . Define  $\mu$  on the degenerate domain  $H_0$  as follows:  $\mu(h_0) = u''$ ; extend  $\hat{w}$  to the domain  $H_0 \cup H_1$

as follows:  $\widehat{w}(u'') = w''$ , and  $\widehat{w}(u) = w'(u)$  for  $u \in U \setminus \{u''\}$ . Clearly, this satisfies properties (i) and (ii) for  $\tau = 0$

Now we complete the definitions of the functions  $\mu$  and  $\widehat{w}$  through induction on  $t$ . Fix  $t > 0$ , and assume we've defined the functions  $\mu : \cup_{\tau=0}^{t-1} H_\tau \rightarrow U$  and  $\widehat{w} : \cup_{\tau=0}^t H_\tau \rightarrow W$  satisfying properties (i) and (ii) for  $\tau = 0, \dots, t-1$  and all  $h_\tau \in H_\tau$ . Consider any  $h_t \in H_t$ . Since  $\widehat{w}(h_t) \in W$ , we know that  $\exists u''(h_t) \in U$  and  $w''(h_t) \in W$  such that  $\widehat{w}(h_t) = (1-\delta)u''(h_t) + \delta w''(h_t)$  and,  $\forall u \in U$ , there exists  $w'(h_t, u) \in W$  with  $\widehat{w}(h_t)R((1-\delta)u + \delta w'(h_t, u))$ . Extend  $\mu$  to the domain  $\cup_{\tau=0}^t H_\tau$  by defining  $\mu(h_t) = u''(h_t)$ , and extend  $\widehat{w}$  to the domain  $\cup_{\tau=0}^{t+1} H_\tau$  by defining  $\widehat{w}(h_t, u)$  as follows:  $\widehat{w}(h_t, u''(h_t)) = w''(h_t)$ , and  $\widehat{w}(h_t, u) = w'(h_t, u)$  for  $u \in U \setminus \{u''(h_t)\}$ . This completes the induction step. Note that, by construction, the functions satisfy properties (i) and (ii) for  $\tau = 0, \dots, t$  and  $h_\tau \in H_\tau$ .

By property (i),  $(1-\delta)C^\mu(h_t)D = \widehat{w}(h_t)$  (that is, the policy program  $\mu$  generates the normalized payoff vector  $\widehat{w}(h_t)$  when initialized at  $h_t$ ). Note in particular that  $(1-\delta)C^\mu(h_0)D = w$ , which means the policy program delivers the desired normalized payoff vector. To complete the argument, we need only show that  $\mu$  is a DCW. Consider any  $h_t$ . Since  $(1-\delta)C^\mu(h_t, u)D = \widehat{w}(h_t, u)$  for all  $u \in U$ , property (ii) immediately implies  $C^\mu(h_t)R^\delta(u, C^\mu(h_t, u))$  for all  $u \in U$ . Thus,  $\mu$  is a DCW.

(b) Consider any  $w \in W$ . We will construct a function  $\mu : \cup_{\tau=0}^\infty H_\tau \rightarrow U$  such that  $\mu(h_0) = w$ , and such that for all  $\tau \geq 0$ ,  $h_\tau \in H_\tau$ , we have (i)  $\mu(h_t) = \mu(h_t, \mu(h_t))$ , and (ii) for all  $u \in U$ , we have  $\mu(h_\tau)R((1-\delta)u + \delta\mu(h_\tau, u))$ .

Since  $w \in \Phi_\delta(W)$ , we know that,  $\forall u \in U$ , there exists  $w'(u) \in W$  with  $wR((1-\delta)u + \delta w'(u))$ . Extend  $\mu$  to the domain  $H_0 \cup H_1$  as follows:  $\mu(w) = w$ , and  $\mu(u) = w'(u)$  for  $u \in U \setminus \{w\}$ . Clearly, this satisfies properties (i) and (ii) for  $\tau = 0$ .

Now we complete the definition of the function  $\mu$  through induction on  $t$ . Fix  $t > 0$ , and assume we've defined the function  $\mu : \cup_{\tau=0}^{t-1} H_\tau \rightarrow U$  satisfying properties (i) and (ii) for  $\tau = 0, \dots, t-1$  and all  $h_\tau \in H_\tau$ . Consider any  $h_t \in H_t$ . Since  $\mu(h_t) \in W$ , we know that,  $\forall u \in U$ , there exists  $w'(h_t, u) \in W$  with  $\mu(h_t)R((1-\delta)u + \delta w'(h_t, u))$ . Extend  $\mu$  to the domain  $\cup_{\tau=0}^{t+1} H_\tau$  by defining  $\mu(h_t, u)$  as follows:  $\mu(h_t, \mu(h_t)) = \mu(h_t)$ , and  $\mu(h_t, u) = w'(h_t, u)$  for  $u \in U \setminus \{\mu(h_t)\}$ . This completes the induction step. Note that, by construction, the function satisfies properties (i) and (ii) for  $\tau = 0, \dots, t$  and  $h_\tau \in H_\tau$ .

By property (i),  $\mu$  is stationary, which means that, for the history  $h_t$ , it delivers

the normalized payoff  $(1 - \delta)C^\mu(h_t)D = \mu(h_t)$ . In particular,  $(1 - \delta)C^\mu(h_0)D = w$ , as required. To complete the argument, we need only show that  $\mu$  is a DCW. Consider any  $h_t$ . Since  $(1 - \delta)C^\mu(h_t, u)D = \mu(h_t, u)$  for all  $u \in U$ , property (ii) immediately implies  $C^\mu(h_t)R^\delta(u, C^\mu(h_t, u))$  for all  $u \in U$ . Thus,  $\mu$  is a DCW. ■

Henceforth, for any policy program  $\mu$ , define

$$\Pi^\mu = \{u \mid u = (1 - \delta)C^\mu(h_t)D \text{ for some feasible history } h_t\}$$

In words,  $\Pi^\mu$  is the collection of normalized payoffs associated with all histories for the policy program  $\mu$ .

**Lemma 6:** *Assume  $U$  is compact. (a)  $\Psi_\delta^\infty$  is identical to the set of normalized payoffs associated with DCWs ( $V_\delta^*$ ). (b)  $\Phi_\delta^\infty$  is identical to the set of normalized payoffs associated with stationary DCWs ( $W_\delta^*$ ).*

**Proof:** (a) First we argue that  $\Psi_\delta^\infty \subseteq V_\delta^*$ . From Lemma 4, part (a), we know that  $\Psi_\delta^\infty = \Psi_\delta(\Psi_\delta^\infty)$ . The desired conclusion then follows from part (a) of Lemma 5.

Next we argue that  $V_\delta^* \subseteq \Psi_\delta^\infty$ . Consider any DCW,  $\mu$ . We claim that  $\Pi^\mu \subseteq \Psi_\delta(\Pi^\mu)$ . From Lemma 2 part (c), it then follows that  $\Pi^\mu \subseteq \Psi_\delta^\infty$ . Since this is true for every DCW  $\mu$ , the desired conclusion is immediate.

To establish the claim, consider any  $w \in \Pi^\mu$ . Let  $h_t$  be some  $t$ -history for which  $(1 - \delta)C^\mu(h_t)D = w$ . Clearly,  $w = (1 - \delta)\mu(h_t) + \delta(1 - \delta)C^\mu(h_t, \mu(h_t))D$ , where  $\mu(h_t) \in U$  and  $(1 - \delta)C^\mu(h_t, \mu(h_t))D \in \Pi^\mu$ . Since  $\mu$  is a DCW, we know that, for all  $u \in U$ , we have  $[(1 - \delta)C^\mu(h_t)D] R [(1 - \delta)u + \delta(1 - \delta)C^\mu(h_t, u)D]$ , where  $(1 - \delta)C^\mu(h_t, u)D \in \Pi^\mu$ . But this implies  $w \in \Psi_\delta(\Pi^\mu)$ , as required.

(b) First we argue that  $\Phi_\delta^\infty \subseteq W_\delta^*$ . From Lemma 4, part (b), we know that  $\Phi_\delta^\infty = \Phi_\delta(\Phi_\delta^\infty)$ . The desired conclusion then follows from part (b) of Lemma 5.

Next we argue that  $W_\delta^* \subseteq \Phi_\delta^\infty$ . Consider any stationary DCW,  $\mu$ . We claim that  $\Pi^\mu \subseteq \Phi_\delta(\Pi^\mu)$ . From Lemma 2 part (d), it then follows that  $\Pi^\mu \subseteq \Phi_\delta^\infty$ . Since this is true for every stationary DCW  $\mu$ , the desired conclusion is immediate.

To establish the claim, consider any  $w \in \Pi^\mu$ . Let  $h_t$  be some  $t$ -history for which  $\mu(h_t) = w$ . Since  $\mu$  is a stationary DCW, we know that, for all  $u \in U$ , we have  $\mu(h_t)R((1 - \delta)u + \delta\mu(h_t, u))$ , where  $\mu(h_t, u) \in \Pi^\mu$ . But this implies  $w \in \Phi_\delta(\Pi^\mu)$ , as required. ■

**Lemma 7:** *Assume  $U$  is compact. (a) The set of normalized payoffs associated with DCWs ( $V_\delta^*$ ) is compact and identical to  $\Phi_\delta^\infty$ . (b) The set of normalized payoffs associated with stationary DCWs ( $W_\delta^*$ ) is compact and identical to  $\Phi_\delta^\infty$ .*

**Proof:** The result follows directly from Lemma 3 and Lemma 6. ■

Lemma 7 is useful because it provides both a tool for proving other results, and an algorithm for finding the set of normalized payoffs associated with DCWs in parameterized applications.

### Proof of Theorem 1

We claim that, when  $U$  is compact and convex,  $V_\delta^* \subseteq \Phi_\delta^\infty$ . Since we know that  $\Phi_\delta^\infty = W_\delta^* \subseteq V_\delta^*$  (Lemma 6, part (b)), the theorem follows directly.

Consider any DCW,  $\mu$ . We establish first that  $\Pi^\mu$  lies in the domain of  $\Phi_\delta$ : with  $U$  convex, we have  $U = \Omega_\delta = \text{Co}(U)$ , so  $\Pi^\mu \subseteq U$  follows from  $\Pi^\mu \subseteq \Omega_\delta$ . The claim then follows from Lemma 5, part (b), as long as  $\Pi^\mu \subseteq \Phi_\delta(\Pi^\mu)$ .

Consider any  $w \in \Pi^\mu$ . Let  $h_t$  be some  $t$ -history for which  $(1 - \delta)C^\mu(h_t)D = w$ . Since  $\mu$  is a DCW, we know that, for all  $u \in U$ , we have  $wR((1 - \delta)u + \delta(1 - \delta)C^\mu(h_t, u)D)$ , where  $(1 - \delta)C^\mu(h_t, u)D \in \Pi^\mu$ . But this implies  $w \in \Phi_\delta(\Pi^\mu)$ , as required. ■

### Proof of Theorem 2

Let  $U$  be the feasible utility set, and let  $H_t^U \equiv U^{t-1}$  be the set of feasible  $t$ -histories, when randomizations are not allowed. With randomizations,  $\text{Co}(U)$  is the feasible utility set, and  $H_t^{\text{Co}(U)} = [\text{Co}(U)]^{t-1}$  is the set of feasible  $t$ -histories.

Consider any policy program  $\mu : \cup_{k=0}^\infty H_k^U \rightarrow U$  that constitutes a DCW when randomizations are not allowed. We construct a policy program  $\hat{\mu} : \cup_{k=0}^\infty H_k^{\text{Co}(U)} \rightarrow U \subseteq \text{Co}(U)$  as follows. For any  $u \in \text{Co}(U)$ , there exists a Borel probability measure  $\lambda(u)$  on  $X$  such that  $E_{\lambda(u)}[\eta(x)] = u$  (where  $E_\lambda$  is the expectations operator given  $\lambda$ ). Let  $\xi(u) = \eta[E_{\lambda(u)}(x)]$ . Since  $X$  is convex,  $E_{\lambda(u)}(x) \in X$ , so  $\xi(u) \in U$ . Since  $\eta_i$  is concave,  $\xi_i(u) \geq u_i$ . Now consider any  $t$  and  $h_t = (u^1, u^2, \dots, u^{t-1}) \in H_t^{\text{Co}(U)}$ . Since  $\hat{\xi}(h_t) \equiv (\xi(u^1), \xi(u^2), \dots, \xi(u^{t-1})) \in H_t^U$ , we can define  $\hat{\mu}(h_t) = \mu(\hat{\xi}(h_t))$ .

To complete the proof, we check that  $\hat{\mu}$  is a DCW when randomizations are allowed. Consider any  $t$ ,  $h_t = (u^1, u^2, \dots, u^{t-1}) \in H_t^{\text{Co}(U)}$ , and  $u \in \text{Co}(U)$ . By construction,  $C^{\hat{\mu}}(h_t) = C^\mu(\hat{\xi}(h_t))$ . Since  $\mu$  is a DCW,  $C^\mu(\hat{\xi}(h_t))R^\delta(\xi(u), C^\mu(\hat{\xi}(h_t), \xi(u)))$ . But since  $\xi_i(u) \geq u_i$  for all  $i$ , and since  $C^\mu(\hat{\xi}(h_t), \xi(u)) = C^{\hat{\mu}}(h_t, u)$  by construction, we have  $C^{\hat{\mu}}(h_t)R^\delta(u, C^{\hat{\mu}}(h_t, u))$ . But this implies that  $\hat{\mu}$  is a DCW. ■

### Proof of Theorem 3

*Part (i):* Proof appears in the text.

*Part (ii):* Define  $\chi(k) \equiv k + 1$  for  $k = 1, \dots, K - 1$ , and  $\chi(K) = 1$ . Choose any  $w^k \in W$ . Since  $w^k P w^{\chi(k)}$ , there plainly exists  $\delta_k^* \in [0, 1)$  such that, for  $\delta \in (\delta_k^*, 1]$ , we have  $w^k P ((1 - \delta)\bar{u} + \delta w^{\chi(k)})$ , which in turn implies  $w^k P ((1 - \delta)u + \delta w^{\chi(k)})$  for all  $u \in U$ . Since  $w^{\chi(k)} \in W$ , this means  $w^k \in \Phi_\delta(W)$ . Taking  $\delta^* = \max_{k=1, \dots, K} \delta_k^*$ , we plainly have  $W \subseteq \Phi_\delta(W)$  for all  $\delta \in (\delta^*, 1]$ . The result then follows as an application of Lemma 5, part (b).<sup>24</sup>

### Proof of Theorem 4

Since  $U$  is convex,  $V_\delta^* = W_\delta^*$  (Theorem 1). Assume, contrary to the theorem, that  $S \cap W_\delta^*$  is non-empty. In light of (i) and the continuity of the majority preference relation, we can represent  $R$  on  $\text{clos}(S)$  by a continuous “utility” function  $\psi(u)$  (Mas-Colell, Whinston, and Green [1995], Proposition 3.C.1, p. 47). Define  $\Theta \equiv \text{clos}(S) \cap W_\delta^*$ . By Lemma 3, part (d),  $\Theta$  is compact. Thus,  $\psi(u)$  reaches a minimum on  $\Theta$ . Choose any  $w^0 \in \arg \min_{u \in \Theta} \psi(u)$ . Note that  $w^0 \in S$  (if not, then since  $S \cap W_\delta^*$  is non-empty by hypothesis, there would exist some  $w \in S \cap W_\delta^*$  such that  $\psi(w) \geq \min \psi(u) = \psi(w^0)$ , which implies  $w R w^0$ ; since  $w^0 \notin S$  by assumption, this violates (ii)).

Now consider some  $u \in U \setminus S$ . Since  $w^0 \in W_\delta^* = \Phi_\delta(W_\delta^*)$ , there exists  $w' \in W_\delta^*$  such that  $w^0 R (1 - \delta)u + \delta w' \equiv v$ . By convexity,  $v \in U$ , and by (ii),  $v \in S$ . There are two cases to consider.

*Case 1:*  $w' \in U \setminus S$ . Since  $v$  is a convex combination of  $w'$  and  $u$ , we must either have  $v R u$  (which occurs if  $w' R u$ ) or  $v R w'$  (which occurs if  $u R w'$ ). But since  $v \in S$ , while  $u, w' \in U \setminus S$ , this contradicts (ii).

*Case 2:*  $w' \in S$ . Since  $u \in U \setminus S$ , (ii) implies  $u P w'$ , which in turn implies  $v P w'$ . Combining this with  $w^0 R v$  yields  $w^0 P w'$  (recall that  $w^0, v, w' \in S$ , on which  $R$  is transitive). But since  $w' \in \Theta$ , this contradicts the definition of  $w^0$ . ■

### Proof of Theorem 5

For any  $w, u \in U$ , define  $\alpha(w, u)$  as follows: if  $N$  is odd,  $\alpha(w, u)$  is the  $M$ -th largest value of  $u_i - w_i$  (among  $i = 1, \dots, N$ ); if  $M$  is even, it is the  $M + 1$ -th largest

<sup>24</sup>It is also possible to prove this result through a direct constructive argument. The proof given here is considerably simpler, however, given that we’ve already proven Lemma 5, part (b) (using a constructive argument that covers a wider range of cases).

value. Notice that  $\alpha$  is continuous. Define  $\alpha^*(w) = \max_{u \in U} \alpha(w, u)$  and select a function  $\hat{u}(w) \in \arg \max_{u \in U} \alpha(w, u)$ . Since  $U$  is compact and  $\alpha$  is continuous, existence of a maximum is assured, so  $\alpha^*$  is well-defined. Moreover, the maximum theorem tells us that  $\alpha^*(w)$  is continuous. Consider  $w^* \in \arg \min_{w \in U} \alpha^*(w)$ . Since  $U$  is compact and  $\alpha^*$  is continuous, existence of a minimum is assured. Since  $w^*$  is not a Condorcet winner by assumption,  $\alpha^*(w^*) > 0$ .

Select a vector  $\bar{u}$  such that  $u \leq \bar{u}$  for  $u \in U$ , and a vector  $\underline{u}$  such that  $u \geq \underline{u}$  for all  $u \in U$  (existence is assured for  $\underline{u}$  and  $\bar{u}$  since  $U$  is compact).<sup>25</sup> Notice that there exists some  $\delta^* > 0$  such that, when  $\delta < \delta^*$ , every element of the vector  $\frac{\delta}{1-\delta}(\bar{u} - \underline{u})$  is strictly less than  $\alpha^*(w^*)$ .

We claim that no DCW exists for  $\delta < \delta^*$ . Consider any policy program  $\mu$ . Discounted payoffs as of period 0 are no greater than  $\mu(h_0) + \frac{\delta}{1-\delta}\bar{u}$ . If the outcome in period 0 is  $\hat{u}(\mu(h_0))$  rather than  $\mu(h_0)$ , discounted payoffs are at least  $\hat{u}(\mu(h_0)) + \frac{\delta}{1-\delta}\underline{u}$ . But with every element of  $\frac{\delta}{1-\delta}(\bar{u} - \underline{u})$  strictly less than  $\alpha^*(w^*)$ , we have  $\left(\hat{u}(\mu(h_0)) + \frac{\delta}{1-\delta}\underline{u}\right) P \left(\mu(h_0) + \frac{\delta}{1-\delta}\bar{u}\right)$ . This means that  $\hat{u}(\mu(h_0))$  and its prescribed continuation is strictly majority preferred to  $\mu(h_0)$  and its prescribed continuation, so  $\mu$  is not a DCW. ■

### Proof of Theorem 6

Consider any  $W \subseteq U$ . We claim that  $\Phi_\delta(W) \subseteq \Phi_{\delta'}(W)$ . Select any  $w \notin \Phi_{\delta'}(W)$ . Then there exists  $u \in U$  such that,  $\forall w' \in W$ ,  $((1-\delta')u + \delta'w') P w$ . Define  $u' = \lambda u + (1-\lambda)w$ , where

$$\lambda \equiv \left(\frac{1-\delta'}{\delta'}\right) \left(\frac{\delta}{1-\delta}\right) \in [0, 1)$$

Since  $U$  is convex,  $u' \in U$ . Through some simple algebra, one can verify that, for any  $w' \in W$ ,

$$(1-\delta)u' + \delta w' - w = \left(\frac{\delta}{\delta'}\right) [(1-\delta')u + \delta'w' - w]$$

But this implies that  $(1-\delta)u'_i + \delta w'_i > w_i$  iff  $(1-\delta')u'_i + \delta w'_i > w_i$ . Consequently,  $\forall w' \in W$ , we have  $((1-\delta)u' + \delta w') P w$ . This is equivalent to  $w \notin \Phi_\delta(W)$ , from which the claim follows directly.

<sup>25</sup>Throughout,  $v \geq w$  denotes  $v_i \geq w_i$  for all  $i$ ,  $v \geq w$  denotes  $v_i \geq w_i$  for all  $i$  with strict inequality for some  $i$ , and  $v > w$  denotes  $v_i > w_i$  for all  $i$ .

Next we argue that  $\Phi_\delta^t \subseteq \Phi_{\delta'}^t$  for all  $t$ . The preceding claim covers the case of  $t = 1$  (taking  $W = U$ ). Now suppose that the property holds for  $t - 1$ . By Lemma 2 part (b),  $\Phi_\delta^t = \Phi_\delta(\Phi_\delta^{t-1}) \subseteq \Phi_\delta(\Phi_{\delta'}^{t-1})$ . Moreover, by the preceding claim,  $\Phi_\delta(\Phi_{\delta'}^{t-1}) \subseteq \Phi_{\delta'}(\Phi_{\delta'}^{t-1}) = \Phi_{\delta'}^t$ . Combining these two statements yields the desired conclusion. Applying induction establishes  $\Phi_\delta^t \subseteq \Phi_{\delta'}^t$  for all  $t$ , from which  $V_\delta^* \subseteq V_{\delta'}^*$  follows directly as an application of Lemma 6, part (b), and Theorem 1. ■

### Proof of Theorem 7

Consider any set  $W \subset U$  and any  $w, v \in W$  with  $v \geq w$ . We claim that  $w \in \Phi_\delta(W)$  implies  $v \in \Phi_\delta(W)$ . This follows immediately from the observation that, for all  $u \in U$  and  $w' \in W$ , we have  $wR((1 - \delta)u + \delta w')$  implies  $vR(1 - \delta)u + \delta w'$ .

Let  $f(w) \equiv \{v \in F(U) \mid v \geq w\}$ . Consider any  $w \in W_\delta^*$ . We know that  $w \in \Phi_\delta^t$  for all  $t$  (Lemma 6, part (b)). Through induction, we will prove that  $f(w) \subseteq \Phi_\delta^t$  for all  $t$ . Consider first  $t = 1$ . Since  $w \in \Phi_\delta^1 = \Phi_\delta(U)$ , the preceding claim implies  $v \in \Phi_\delta(U) = \Phi_\delta^1$  for all  $v \in f(w) \subseteq U$ , as desired. Now assume the statement is true for  $t - 1$ . Since  $w \in \Phi_\delta^t = \Phi_\delta(\Phi_\delta^{t-1})$ , the preceding claim likewise implies that  $v \in \Phi_\delta(\Phi_\delta^{t-1}) = \Phi_\delta^t$  for all  $v \in f(w) \subseteq \Phi_\delta^{t-1}$ , as desired. Thus, applying induction along with Lemma 6, part (b), we have  $f(w) \subseteq W_\delta^*$ .

Now consider any  $w \in W_\delta^*$  with  $w \notin F(U)$ . Since  $W_\delta^* \subseteq U$ , there exists some  $v \in f(w)$  such that  $v > w$ . Since  $f(w) \subseteq W_\delta^*$ , this means  $w \notin F(W_\delta^*)$ . Similarly, consider any  $w \in W_\delta^*$  with  $w \notin F^s(U)$ . Since  $W_\delta^* \subseteq U$ , there exists some  $v \in f(w)$  such that  $v \geq w$ . Since  $f(w) \subseteq W_\delta^*$ , this means  $w \notin F^s(W_\delta^*)$ . The theorem then follows from the fact that  $V_\delta^* = W_\delta^*$  when  $U$  is compact and convex. ■

### Proof of Theorem 8

We provide a separate argument for each part of the theorem. Throughout, recall that, since  $U$  is convex,  $U = \text{Co}(U)$ . For the purpose of this proof, we need to make certain functional arguments, previously suppressed, explicit:  $\Phi_\delta(\bullet, A)$  refers to the self-generation map when the feasible payoff space is  $A$  (so far, we have written  $\Phi_\delta(\bullet, U)$  as  $\Phi_\delta(\bullet)$ );  $\Phi_\delta^t(A)$  is obtained through  $t$  successive applications  $\Phi_\delta(\bullet, A)$ , starting with  $A$  (so far, we have written  $\Phi_\delta^t(U)$  as  $\Phi_\delta^t$ ).

*Part (i):* The proof of part (i) proceeds through a series of three steps.

*Step 1:*  $E_\delta \subseteq F(V_\delta^*)$ . Suppose  $w \in E_\delta$ . Then, by definition,  $w \in F(U)$ , and  $w \in V_\delta^*$ . Since  $V_\delta^* \subseteq U$ , we have  $w \in F(V_\delta^*)$ .

*Step 2:*  $F(V_\delta^*) \subseteq W_\delta^*(F(U))$ .

By Lemma 6, part (b),  $W_\delta^*(F(U)) = \bigcap_{t=1}^\infty \Phi_\delta^t(F(U))$ . To prove the desired result, we first establish the following property.

**Lemma 8:** *For any  $W \subseteq U$ , we have  $F[\Phi_\delta(W, U)] \subseteq \Phi_\delta(F(W), F(U))$ .*

**Proof:** Consider any  $w \in F[\Phi_\delta(W, U)]$ . From the argument given in the proof of Theorem 7, we know that  $w \in F(W)$ . Since  $F(U) \subseteq U$ , we also know that,  $\forall u \in F(U)$ ,  $\exists w' \in W$  such that  $wR((1-\delta)u + \delta w')$ . Take any such  $u$  and the associated  $w'$ . Select any  $j$  such that at least  $M$  agents other than  $j$  weakly prefer  $w$  to  $(1-\delta)u + \delta w'$ . Define  $H = \{\tilde{w} \in W \mid \tilde{w}_i = w'_i \text{ for } i \neq j\}$ , and let  $w'' = \arg \max_{\tilde{w} \in H} \tilde{w}_j$ .

We claim that  $w'' \in F(W)$ . Suppose not. Then  $\exists v \in W$  such that  $v > w''$ . Define  $v'$  as follows:  $v'_j = v_j$ , and  $v'_i = w''_i$  for  $i \neq j$ . By free disposal,  $v' \in W$ , and by construction  $v' \in H$ . But since  $v'_j > w''_j$ , this contradicts the definition of  $w''$ .

Since  $wR((1-\delta)u + \delta w')$ , and since only the  $j$ -th elements of  $w'$  and  $w''$  differ, we have  $wR((1-\delta)u + \delta w'')$ . But this implies  $w \in \Phi_\delta(F(W), F(U))$ , as required. ■

Now we continue with the proof of step 2. By Lemma 8, we know that

$$F(\Phi_\delta^1(U)) = F(\Phi_\delta(U, U)) \subseteq \Phi_\delta(F(U), F(U)) = \Phi_\delta^1(F(U))$$

Now assume that  $F(\Phi_\delta^{t-1}(U)) \subseteq \Phi_\delta^{t-1}(F(U))$ . Then

$$\begin{aligned} F(\Phi_\delta^t(U)) &= F(\Phi_\delta(\Phi_\delta^{t-1}(U), U)) \\ &\subseteq \Phi_\delta(F(\Phi_\delta^{t-1}(U)), F(U)) \\ &\subseteq \Phi_\delta(\Phi_\delta^{t-1}(F(U)), F(U)) \\ &= \Phi_\delta^t(F(U)) \end{aligned}$$

Thus, by induction,  $\bigcap_{t=1}^\infty F(\Phi_\delta^t(U)) \subseteq \bigcap_{t=1}^\infty \Phi_\delta^t(F(U)) = W_\delta^*(F(U))$  (where the equality follows from Lemma 6, part (b)).

Next we claim that  $F(W_\delta^*(U)) \subseteq \bigcap_{t=1}^\infty F(\Phi_\delta^t(U))$ . Combining this with the conclusion in the previous line gives us  $F(W_\delta^*(U)) \subseteq W_\delta^*(F(U))$ ; in light of Theorem 1, this implies  $F(V_\delta^*) \subseteq W_\delta^*(F(U))$ , which is what we set out to prove.

To establish the claim, suppose that  $w \in F(W_\delta^*(U))$ . Then  $w \in \Phi_\delta^t(U) \subseteq U$  for all  $t$ , and  $w \in F(U)$  (by Theorem 7). Consequently,  $w \in F(\Phi_\delta^t(U))$  for all  $t$ , which implies  $w \in \bigcap_{t=1}^\infty F(\Phi_\delta^t(U))$ .

*Step 3:*  $W_\delta^*(F(U)) \subseteq E_\delta$ . For any  $w \in W_\delta^*(F(U))$ , there exists a stationary DCW  $\mu$  such that  $\mu(h_0) = w$  and  $\mu(h_t) \in F(U)$  for all  $h_t$ , given the utility possibility set  $F(U)$ . We define another stationary policy program  $\mu'$  for the utility possibility set  $U$  as follows. Consider any function  $z(u)$  such that, for all  $u \in U \setminus F(U)$ ,  $z(u) \geq u$ , and  $z(u) \in F(U)$ ; for all  $u \in F(U)$ ,  $z(u) = u$ . For any  $t$ -history  $h_t = (u^1, \dots, u^{t-1})$ , let  $Z(h_t) = (z(u^1), \dots, z(u^{t-1}))$ , and let  $\mu'(h_t) = \mu(Z(h_t))$ . Plainly,  $C^\mu(h_0) = (w, w, \dots)$ , and the associated normalized payoff is  $w$ . Moreover, for any  $h_t$  and  $u \in U$ , we know by construction that  $C^{\mu'}(h_t) = C^\mu(Z(h_t))$ , that  $C^{\mu'}(h_t, u) = C^\mu(Z(h_t), z(u))$ , that  $z(u)$  weakly Pareto dominates  $u$ , and that  $C^\mu(Z(h_t))R^\delta(z(u), C^\mu(Z(h_t), z(u)))$  (since  $\mu$  is a DCW). Putting these facts together gives us  $C^{\mu'}(h_t)R^\delta(u, C^{\mu'}(h_t, u))$ , which means  $\mu'$  is a DCW when the feasible payoff set is  $U$ . By construction, it is also UWE.

Plainly, steps 1-3 together imply that  $E_\delta = F(W_\delta^*) = W_\delta^*(F(U))$ .

*Part (ii):* First we claim that, with strict comprehensiveness,  $F^s(U) = F(U)$ . Clearly,  $w \in F^s(U)$  implies  $w \in F(U)$ . So assume that  $w \in F(U)$ . Suppose  $w \notin F^s(U)$ . Then there exists some  $w' \in U$  such that  $w' \geq w$ . But then there must also exist some  $w'' \in U$  with  $w'' > w$ , which contradicts  $w \in F(U)$  and thereby establishes the claim. It follows immediately that  $E_\delta = E_\delta^s$ , and  $W_\delta^*(F(U)) = W_\delta^*(F^s(U))$ .

To complete the proof, we need to show that  $F(W_\delta^*(U)) = F^s(W_\delta^*(U))$  (which implies  $F(W_\delta^*) = F^s(W_\delta^*)$  by Theorem 1). Obviously,  $F^s(W_\delta^*(U)) \subseteq F(W_\delta^*(U))$ . From Theorem 7, we know that  $F(W_\delta^*(U)) \subseteq F(U)$ , and we have just shown that  $F(U) = F^s(U)$  with strict comprehensiveness, so  $F(W_\delta^*(U)) \subseteq F^s(U)$ . Thus, for any  $w \in F(W_\delta^*(U))$  and  $w' \in U$  with  $w' \neq w$ , we have  $w_i > w'_i$  for some  $i$ . Since  $W_\delta^*(U) \subseteq U$ , this implies that, for any  $w \in F(W_\delta^*(U))$  and  $w' \in W_\delta^*(U)$  with  $w' \neq w$ , we have  $w_i > w'_i$  for some  $i$ . But then  $w \in F^s(W_\delta^*(U))$ , as desired. ■

### Proof of Theorem 9

Let  $\Lambda$  denote the set of Borel probability measures on  $X$ . For each  $i$ , define  $\theta_i : \Lambda \rightarrow \mathbb{R}$  as follows:  $\theta_i(\lambda) = E_\lambda[\eta_i(x)]$  (where  $E_\lambda$  denotes the expectation with respect to the probability measure  $\lambda$ ). The utility possibility set is then given by  $U = \theta(\Lambda)$ . Standard arguments imply that  $U$  is compact and convex.

We claim first that  $F(U) \subseteq \eta(X)$ . Consider any  $\lambda \in \Lambda$  such that there is no  $x \in X$  with  $\theta(\lambda) = \eta(x)$ . Then there exists some open set  $Q \subset X$  with

$\lambda(Q) \in (0, 1)$ . Since  $\eta_i$  is strictly concave,  $\theta_i(\lambda) = E_\lambda[\eta_i(x)] < \eta_i(E_\lambda[x])$ . But this implies  $\theta_i(\lambda) \notin F(U)$ , as claimed.

Since  $U$  is compact, so is  $F(U)$ . Lemma 6, part (b) therefore implies that  $W_\delta^*(F(U))$  is compact. Consequently, we can define  $w^0 = \arg \min_{w \in W_\delta^*(F(U))} w_M$ . Since  $F(U) \subseteq \eta(X)$ , we know that  $w^0 = \eta(y^0)$  for some  $y^0 \in [0, 1]$ .

Now assume that  $y^0 \neq y_M$ . Without loss of generality, assume that  $y^0 > y_M$ . By Lemma 4, part (b), we know there exists  $w' \in W_\delta^*(F(U))$  such that  $w^0 R(\delta w^C + (1 - \delta)w')$ . Since  $W_\delta^*(F(U)) \subseteq F(U) \subseteq \eta(X)$ , we know that  $w' = \eta(y')$  for some  $y' \in [0, 1]$ , and we also know that  $\eta_M(y') \geq \eta_M(y^0)$  (by definition of  $y^0$ ), which in turn implies  $y' \leq y^0$ . But then, by the single crossing property, we have  $w'_i = \eta_i(y') \geq \eta_i(y^0) = w_i^0$  for  $i \in \{1, \dots, M\}$ . Since all  $i \in \{1, \dots, M\}$  also strictly prefer  $w^C \equiv \eta(y_M)$  to  $w^0$ , this contradicts  $w^0 R(\delta w^C + (1 - \delta)w')$ .

It follows that  $y^0 = y_M$ , which implies  $W_\delta^*(F(U)) = \{\eta(y_M)\}$ . Since  $U$  is compact and convex, Theorem 8 implies that  $E_\delta = W_\delta^*(F(U)) = \{\eta(y_M)\}$ . ■

### Proof of Theorem 10

Since  $U$  is convex and compact,  $V_\delta^* = W_\delta^*$  (Theorem 1), which allows us to focus attention on stationary DCWs.

*Part (i):* Suppose not. Without loss of generality, consider  $w \in W_\delta^*$  with  $\sum_{i=1}^M w_i < 1 - \delta$ . Consider any  $u \in U$  and some small  $\varepsilon > 0$  with  $u_i = \frac{w_i}{1 - \delta} + \varepsilon$  for  $i = 1, \dots, M$  (choosing  $\varepsilon$  sufficiently small guarantees  $\sum_{i=1}^M u_i < 1$ ). Note that, for  $i = 1, \dots, M$ , we have  $w_i < (1 - \delta)u_i + \delta w'_i$  for all  $w' \in U$  (since  $w_i \geq 0$ ). Consequently,  $w \notin \Phi_\delta(U) \supseteq W_\delta^*$ .

*Part (ii):* First consider the case where  $\delta < \frac{M-1}{N}$ . With equal division ( $u_i = \frac{1}{N} \forall i$ ), for any coalition  $A$  with  $|A| = M$ , we have  $\sum_{i \in A} u_i = \frac{M}{N}$ . Consequently, for all  $u \in U$ , there exists some coalition  $A$  with  $|A| = M$ , such that  $\sum_{i \in A} u_i \leq \frac{M}{N}$ . But  $\delta < \frac{M-1}{N}$  implies  $\frac{M}{N} < 1 - \delta$ , so  $u \notin \Delta_\delta^N$ . Therefore, by part (i),  $u \notin W_\delta^*$ .

Now consider the case where  $\delta \in [\frac{M-1}{N}, \frac{1}{2})$ . Suppose  $W_\delta^* \neq \emptyset$ . Since  $W_\delta^*$  is compact, we can define  $u_0 \equiv \min_i \min_{u \in W_\delta^*(U)} u_i$ . We claim that all  $w \in W_\delta^*$  satisfy the following property:  $\forall A \subset \{1, \dots, N\}$  with  $|A| = M$ ,  $\sum_{i \in A} w_i \geq 1 - \delta + M\delta u_0$ . Consider any  $w$  such that, for some  $A \subset \{1, \dots, N\}$  with  $|A| = M$ ,  $\sum_{i \in A} w_i < 1 - \delta + M\delta u_0$ . Take any  $u \in U$  and some small  $\varepsilon > 0$  with  $u_i = \frac{w_i - \delta u_0}{1 - \delta} + \varepsilon$  for  $i = 1, \dots, M$  (choosing  $\varepsilon$  sufficiently small guarantees  $\sum_{i=1}^M u_i < 1$ ). Note that, for  $i = 1, \dots, M$ , we have  $w_i < (1 - \delta)u_i + \delta w'_i$  for all  $w' \in U$  (since  $w'_i \geq u_0$ ).

Consequently,  $w \notin \Phi_\delta(U) \supseteq W_\delta^*(U)$ , which establishes the claim.

Now consider  $w \in W_\delta^*$  such that  $w_i = u_0$  for some  $i$ . Without loss of generality, assume  $w_1 = u_0$ . By the preceding claim, for all  $A \subset \{2, \dots, N\}$  with  $|A| = M - 1$ ,  $\sum_{i \in A} w_i \geq 1 - \delta + M\delta u_0 - u_0$ . Consider  $A_1, A_2 \subset \{2, \dots, N\}$  with  $|A_k| = M - 1$  for  $k = 1, 2$ , and  $A_1 \cup A_2 = \{2, \dots, N\}$ . Then

$$\begin{aligned} \sum_{i=1}^N w_i &= w_1 + \sum_{i \in A_1} w_i + \sum_{i \in A_2} w_i \\ &\geq u_0 + 2(1 - \delta + M\delta u_0 - u_0) \\ &= 2(1 - \delta) + (2M\delta - 1)u_0 \end{aligned}$$

Since  $\delta < \frac{1}{2}$ , we have  $2(1 - \delta) > 1$ . Since  $\delta \geq \frac{M-1}{N}$ , we have  $2M\delta - 1 > 0$ . Since  $u_0 \geq 0$ , we have  $\sum_{i=1}^N w_i > 1$ , which implies that  $w$  is infeasible – a contradiction.

*Part (iii):* We claim that, for  $\delta \geq \frac{N-1}{N+1}$ ,  $\Phi_\delta(\Delta_\delta^N) = \Delta_\delta^N$ . Consider any  $w \in \Delta_\delta^N$  and any  $u \in U$ . Define  $z_i = w_i - (1 - \delta)u_i$ . Without loss of generality, assume that the  $z_i$  are weakly decreasing in  $i$ . Note that

$$\sum_{i=1}^N z_i = \sum_{i=1}^N w_i - (1 - \delta) \sum_{i=1}^N u_i = \delta \quad (1)$$

Since the  $z_i$  are arranged in descending order, 1 implies

$$\sum_{i=1}^M z_i \geq \frac{M}{N} \delta \quad (2)$$

In addition, we claim that  $z_i \geq 0$  for  $i = 1, \dots, M$ . Suppose not. Then  $z_i < 0$  for  $i = M, \dots, N$ . But

$$\sum_{i=M}^N z_i = \sum_{i=M}^N (w_i - (1 - \delta)u_i) \geq 0.$$

This is because  $w \in \Delta_\delta^N$  and  $|\{M, \dots, N\}| = M$  imply  $\sum_{i=M}^N w_i \geq 1 - \delta$ , while  $u \in U$  implies  $\sum_{i=M}^N u_i \leq 1$ . Thus we have a contradiction.

Now construct  $w' \in \Delta_\delta^N$  as follows:

$$w'_i = \begin{cases} \frac{z_i}{\sum_{i=1}^M z_i} (1 - \delta) & \text{for } i = 1, \dots, M \\ \frac{\delta}{M-1} & \text{for } i = M + 1, \dots, N \end{cases}$$

Note that  $\sum_{i=1}^N w'_i = 1$  and that  $w'_i \geq 0$  for all  $i$ , so  $w' \in \Delta^N$ . Note also that  $\sum_{i=1}^M w'_i = 1 - \delta$ , and for all other  $A \subset \{1, \dots, N\}$  with  $|A| = M$  we have  $\sum_{i \in A} w'_i \geq$

$w_{M+1} = \frac{\delta}{M-1} \geq \frac{N-1}{(N+1)(M-1)} = 1 - \frac{N-1}{N+1} \geq 1 - \delta$ . Thus,  $w' \in \Delta_\delta^N$ . Finally, note that, for  $i = 1, \dots, M$ ,

$$\begin{aligned} (1 - \delta) u_i + \delta w'_i &= w_i - z_i + \delta w'_i \\ &= w_i + z_i \left( \frac{\delta(1 - \delta)}{\sum_{i=1}^M z_i} - 1 \right) \\ &\leq w_i + z_i \left( \frac{N}{M}(1 - \delta) - 1 \right) \\ &\leq w_i \end{aligned}$$

where the first inequality follows from 2, and the second inequality follows from  $\delta \leq \frac{N-1}{N+1}$ . Thus,  $w \in \Phi_\delta(\Delta_\delta^N)$ .

By lemma 5, part (b),  $\Phi_\delta(\Delta_\delta^N) = \Delta_\delta^N$  implies  $\Delta_\delta^N \subseteq W_\delta^*$ . Combining this with  $W_\delta^* \subseteq \Delta_\delta^N$  (part (i)), we have  $\Delta_\delta^N = W_\delta^*$ . ■

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