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Assignment Messages and Exchanges

By
Paul Milgrom
Stanford University

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Stanford Institute for Economic Policy Research
Stanford University
Stanford, CA 94305
(650) 725-1874

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Paul Milgrom[1]

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Abstract: "Assignment messages" are messages that parameterize substitutable preferences using a particular linear program. An "assignment exchange" is a simplified Walrasian exchange in which participants are restricted to report only assignment messages. Any pure Nash or ε-Nash equilibrium of the simplified mechanism is a Nash or ε-Nash equilibrium of the Walrasian mechanism before simplification. With a further restriction to basic assignment messages, the exchange yields integer-valued allocations, thus generalizing the Shapley-Shubik assignment mechanism. Connections are reported between assignment exchanges and ascending multi-product clock auctions, double auctions for a single product, and Vickrey auctions. Applications include some cases of Leontieff complements.

Keywords: mechanism design, market design, auctions, Shapley-Shubik

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I. Introduction

In abstract mechanism theory, the designer is often presumed able to create a direct mechanism in which each participant's reports its "type," revealing all the details

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of the participant's preferences as well as anything else the participant may know. In practice, these details can be so numerous that reporting them all is impractical.

For example, in the National Resident Matching Program, which places doctors into hospital residency programs (Roth and Peranson (1999)), a hospital that interviews fifty candidates in the hopes of employing ten has a type that is a list of length approximately 1.3×10^{10} ranking all of the subsets of size ten or less. In FCC Spectrum Auction 73, completed in 2008 with the sale of 1,090 radio spectrum licenses, a type is a vector of approximate dimension 1.3×10^{328}, listing values for every subset of licenses. Such reports are far too lengthy to be practical.

There are two pure approaches to mitigating the length-of-report problem. The first is to simplify reporting by limiting the message space and the second uses dynamic reporting. The National Resident Matching Program uses the first strategy: it limits hospitals to report preferences in the form of a number of positions and a rank order listing of candidates. If a hospital has ten openings and interviews fifty candidates, it reports the number ten and a list of length fifty – a manageably small message.

This main innovation of this paper is the introduction and analysis of a new message space – the space of *assignment messages* – designed for practical use. Assignment messages describe preferences indirectly as the value of a linear program with constraints that conform to particular structure. These messages are formulated for mechanism design problems in which participants regard the goods as substitutes. A further simplification to *basic* assignment messages is particularly useful for problems of allocating indivisible, substitutable goods. Such messages demand only integer quantities and limit the local rates of substitution between any two goods to be zero or one. This restriction ensures that an efficient allocation assigns integer quantities of each good to each participant.

This paper analyzes the properties of assignment messages and *assignment exchanges*, which are simplified direct Walrasian mechanisms in which participants may report only assignment messages. *Assignment auctions* are assignment exchanges with just one buyer or just one seller, so that the competitive bids come from just one side of the market. With basic assignment messages, the assignment exchange extends its namesake, the Shapley and Shubik (1972) assignment mechanism, in three main ways: it allows participants to trade multiple types of goods, to trade multiple units of each type of good, and to have dual roles, buying some goods and selling others.

The simplified message approach must be compared to the second pure approach to reducing communication, which eschews direct mechanisms in favor of dynamic ones with staged reporting of information (Nisan and Segal (2006), Parkes (2005)). With stage reporting, only partial information about a participant's type is revealed at each reporting stage. Examples of dynamic multi-product mechanisms include simultaneous ascending and descending auctions (eg, Ausubel (2007)), in which bidders are asked to report supplies or demands at a sequence of announced prices. Since demands at some prices cannot be inferred from the reported information, this economizes on communications relative to a standard direct mechanism.

For the problem of general equilibrium with substitutable preferences, the most relevant dynamic mechanisms are the simultaneous ascending or descending multi-item auctions, which have used in a variety of applications involving radio spectrum, electricity, and natural gas (Milgrom (2004)). When the goods for sale are substitutes and that participants bid myopically, various types of simultaneous ascending or descending auctions not only economize on communications but also, in certain cases, identify efficient or stable allocations or find minimum or maximum market-clearing prices (Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2004), Milgrom and Strulovici (2008)).

Compared to the similar direct mechanisms, multi-product ascending and descending auctions suffer from high participant costs, long times-to-completion, imprecise computations, and difficulties of scheduling. Any multi-round, real-time process adds the cost of real-time bidding to the costs of preparing for the auction. Increasing the frequency of rounds can reduce this time cost, but that both increases the risk of bidder error and reduces any advantage that dynamic mechanisms may have in allowing bidders to respond thoughtfully to emerging information. In current practice, dynamic auctions for gas and electricity take several hours to reach completion, while spectrum auctions take days, weeks, or even months. Such long times-to-completion cripple these mechanisms for the most time-sensitive markets, such as hour-ahead power markets, where only minutes are available to complete an exchange. Real ascending and descending auctions generally fail to identify exact market-clearing prices in finite time, because they change the direction of price increments only a finite number of times and because price increments are discrete.[2] Pricing errors tend to be most severe when there are many products being exchanged, because finding equilibrium prices then requires a time-consuming search of a high-dimensional price-space. Finally, in export markets

[2] Ausubel and Cramton (2004) show how a clock auction with bidder-determined step sizes ("intra-round bidding") can be made to converge asymptotically to an exact solution if goods are divisible and all demand sets are singletons.

where potential buyers may reside in a dozen or more different time zones, scheduling a convenient hour for real-time bidding may be impossible.

These four problems are avoided by direct mechanisms, including simplified direct mechanisms. The key challenge for implementing a usable direct mechanism is to design a simple message space with suitable properties. For some applications, one such property is that the mechanism should lead to integer allocations.

Basic assignment messages result in efficient allocations that are integer valued, but require the marginal rates of technical substitution to be everywhere zero or one. This strong restriction is surprisingly often a reasonable approximation for auction and exchange applications. For example, an electric utility delivering retail power to its customers might acquire wholesale power from generators at different locations, say A, B and C, but may be limited in its ability to utilize power from each source by its source-specific transmission capacities. When sufficient transmission capacity is available at source A, one unit of power from A can be substituted for one unit from any other source. When it is not, an additional unit of power at A displaces zero units of other power from other sources. Similarly, a cereal maker may be able to substitute bushels of grain delivered today for bushels delivered tomorrow up to a limit imposed by its grain storage capacity, or it may substitute one metric ton of a particular type or grade of grain for one of another type within limits specified by production requirements. The same limited one-for-one substitution pattern is sometimes found among sellers as well, as when a manufacturer can deliver several versions of the same processed good in a total amount that is limited by the overall capacity of its factory.[3] Limited one-for-one substitution can be a useful approximation whenever lots differ only in attributes such as time and location of availability, grade, degree of processing, or delivery and contract terms.

By limiting messages to express limited one-for-one substitution, the basic assignment exchange enjoys two advantages compared to the general assignment exchange: messages are simpler and, if the quantities in each bid and constraint group are integers, then the resulting allocations are always integer-valued.[4] The integer allocation property can be important in some applications, such as ones where commodities are most efficiently shipped by the truck- or container-load. Even for goods like electric power, which in principle seem perfectly divisible, contracts are often denominated and traded in indivisible units, such as megawatts of power, so respecting integer constraints may sometimes be useful.[5]

General assignment messages extend the basic assignment messages by allowing participants to specify rates of technical substitution besides zero and one. For example, in markets for electric power, if the transmission losses in shipping power from A are higher than from B, then one unit of power at A displaces less than one unit from B – the rate of technical substitution is less than one, and general assignment messages can express that. Using basic assignment messages, a bidder can account for such

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[3] The National Resident Matching Program, with its fixed number of slots at each hospital, imposes one-for-one substitution but excludes resident wages from the process. An assignment auction could be suitable for that application, provided that wages are made endogenous. Crawford (2004) proposes a simultaneous ascending auction mechanism for the same application.

[4] When bundles necessarily consist of integer quantities and goods are substitutes, a version of the limited one-for-one substitution property is implied (Gul and Stacchetti (1999), Milgrom and Strulovici (2008)).

[5] When the number of units transacted is large, integer allocations often become less important because rounding of fractional allocations may become a viable alternative.

transmission losses only by treating the power from different sources as having different values.

A simplification limits the messages used by a mechanism, and that can affect incentives and performance. In a general simplification, some message profiles may be equilibria of the simplified mechanism even though they were not equilibria in the original, extended mechanism. A *tight* simplification is one with the property that, for every profile of participant preferences, all of the full-information pure ε-Nash equilibria of the simplified mechanism are also ε-Nash equilibria of the original mechanism (see Milgrom (2008)). Assignment exchanges are tight simplifications of general Walrasian exchange mechanisms.

An important attribute of assignment messages is that they allow not only bids to buy or sell one of several different goods, but also "swap" bids. For example, in a securities market, a swap could link an offer to buy shares of stock with an offer to sell certain call options in a single bid group which specifies that the trade should takes place if and only if the net cost per share of the transaction does not exceed a participant-specified maximum. Such a linkage can be valuable, because it eliminates execution risk.[6]

Assignment messages can express directly only substitutable preferences, so it is perhaps surprising that the basic assignment exchange nevertheless has applications to some resource allocation problems involving *complementary* goods, for which package

[6] Some traders call this "leg risk" because the danger is that one "leg" of a transaction is executed while the other is not.

exchange mechanisms might have been thought to be necessary.[7] Figure 1 displays an example.

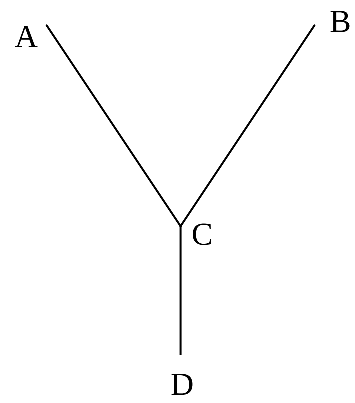

Figure 1: A Y-Shaped Electrical Transmission Grid

Points A, B and C in the figure represent physical locations (in southeast Wyoming) where wind farms produce electrical power carried by new long-range transmission lines, while point D represents a node (in northwest Colorado) where the power is injected into the existing transmission grid. For a producer located at A, transmission capacity along the lines AC and CD are Leontief complements: the producer is constrained by the minimum of the capacity acquired on AC or CD. Similarly, producers at B regard BC and CD as Leontief complements. The power producers located at A, B and C compete to acquire capacity on the CD link. There are also suppliers of capacity on each link. I assume that there are one or more separate capacity suppliers for each link and that if there are any suppliers that can supply more than one link, then their costs of supply are additively separable across links.

[7] See Milgrom (2007) for an introduction to the economic package allocation problem, Nisan (2006) for an analysis of some message spaces that might be used in package auctions, and Cramton, Shoham, and Steinberg (2006) for a collection of related articles.

Despite the technical complementarities among successive links, preferences of both buyers of transmission links and of suppliers of capacity can be expressed using basic assignment messages. The key lies in the way *lots* are defined. Suppose the exchange is organized to trade three kinds of lots. Each lot is a package of links sufficient to transmit a unit of energy from one of the points A, B or C to point D. With lots defined in that way, each energy producer/capacity buyer can bid on the lot connecting its location to the root at D, so these participants can express their preferences accurately. A supplier who wishes to offer capacity on one of the single links AC or BC can do that using a *swap* – linked offers to buy and sell. For example, an offer to sell capacity on AC at a price of at least X is represented as a swap that links an offer to sell capacity on the AD lot with a bid to buy equal capacity on the CD lot at a price difference of at least X. With the specified lots, both buyers and sellers can express preferences accurately. The theorems about assignment exchanges apply and imply that, despite complementarities and indivisible lots which are problematic in other settings, market-clearing prices exist.

A similar construction can be used in any acyclic network by identifying one node in each component of the graph as a root and expressing all lots in terms of flows from a node to a root. Demand need not be located only at the roots for this construction to work, but the demanded packages of links must lie in sequence on one side of the root. A potentially interesting example of distributed demand is in a market for landing slots at a busy airport.

Airlines with hub-and-spoke systems usually prefer to operate groups of landing slots within a short interval of time, so that delays for passengers with connecting flights can be minimized. Suppose that landing slots are discrete and labeled by their assigned times T. To express the airline's preferences using assignment messages, operate the exchange as if the lots in the auction were intervals of slots $(0,T]$, for $T = 1,2,\ldots$, consisting of consecutive slots from the beginning of the day until some last time T. An airline wishing to purchase the successive landing slots in the interval $(T_0,T_1]$ for a price not exceeding P could express that by linking a bid to buy $[0,T_1]$ with an offer to sell $[0,T_0]$. Depending on the prices of each slot, the same airline might be willing to start a bit earlier or later and might wish to buy more or fewer slots. To describe its full preferences, the airline could bid for many time intervals with different numbers of slots, different starting times, and different net prices. If some particular time T is included in all the reported intervals, then at most one swap in the set can be executed. If all airlines' preferences could be similarly expressed, then what might otherwise have been a combinatorial optimization problem for which, in the general case, computation is hard and market-clearing prices may not exist, is replaced by a simple assignment exchange problem, for which a value-maximizing integer solution and supporting prices are easily computed.

The remainder of this paper is organized as follows. Section II introduces the assignment message space and reports three theorems about it. The first is that the assignment messages express only substitutable preferences. The second is that when all preferences are expressed by assignment messages, then the set of market-clearing prices is a non-empty, closed, convex sublattice. The third is that if all participants' preferences are expressed with *basic* assignment messages, then there is an efficient allocation using only integer quantities of all goods. Section III provides a partial converse to the these theorems. Assignment messages require that the constraints connecting different goods form a “tree.” If that constraint is relaxed at all, then the first two conclusions of section II are no longer valid. Section IV is about tightness. Its main conclusion is that the assignment exchanges, as well as many further simplifications of these exchanges, are tight simplifications of a Walrasian mechanism. Section V discusses the connections between the assignment exchange mechanism, single product uniform price auctions, and the Vickrey auction. Section VI concludes with a discussion of steps to take the theory to applications.

II. Assignment Messages

Consider a resource allocation problem with goods indexed by $k = 1,...,K$ and participants are indexed by $n = 1,...,N$. If participants’ preferences are quasi-linear, then the utility for a trade is expressed as the value $V_n(q_n)$ of bundle $q_n \in \mathbb{R}^K$ acquired plus any net cash transfer. The set of demanded bundles at price vector *p* is $\arg\max_{q_n} V_n(q_n) - p \cdot q_n$, where q_n may include both positive and negative components. A direct mechanism must specify a message space for describing V_n. The assignment exchange determines q_n by summing vectors x_j for $j \in J(n)$, where *j* is the serial number of a bid and *J*(*n*) is the set of serial numbers for bids by bidder *n*.

An assignment message describes V_n using a collection of bids and a set of constraints.[8] Each bid by bidder *n* is indexed by a serial number $j \in J(n)$ and consists of a 5-tuple $(k_j, v_j, \rho_j, l_j, u_j)$ where k_j identifies the type of product, v_j identifies the “value” of the bid, $\rho_j > 0$ identifies the “effectiveness” and the remaining two terms are lower

[8] A related precursor to this message space is the space of *endowed assignment messages*, introduced by Hatfield and Milgrom (2005).

and upper bounds on quantities: $l_j \le 0 \le u_j$. The role of the effectiveness coefficient, which is to allow arbitrary marginal rates of substitution within certain constraints, will be formalized shortly.

In addition to the bids, participant *n*'s assignment message expresses constraints of two kinds. First are the *single-product bid group constraints*:

$$l_{kS} \le \sum_{j \in S} x_j \le u_{kS} \quad \text{for } S \in \mathfrak{T}_{nk} \tag{1}$$

where $\mathfrak{T}_{nk}$ includes all singletons $S = \{j\}$ for which $k_j = k$ and may include other subsets of $R_{nk} = \{ j \in J(n) \mid k_j = k\}$. For the singletons, $l_{k_j \{j\}} \equiv l_j$. Second are the *multi-product bid group constraints* indexed by the set $\mathfrak{T}_{n0}$. These are of the form

$$l_{0S} \le \sum_{j \in S} \rho_j x_j \le u_{0S} \quad \text{for } S \in \mathfrak{T}_{n0}\,. \tag{2}$$

Unlike the sets used in the single product group constraints, the sets $S \in \mathfrak{T}_{n0}$ may include bids on multiple products. Also, unlike the sums in (1), those in (2) are weighted sums, with weights equal to the effectiveness coefficients.

To simplify notation, we suppress the subscript *n* while we are analyzing the reports and preferences of a single bidder; the subscript will reappear later when we analyze allocations for multiple participants. Using the bids and constraints, bidder *n*'s message is interpreted to report a value for any feasible bundle of products $q = (q_1, \ldots, q_K)$ as follows:

$$
\begin{aligned}
V(q) = \max_x \textstyle\sum_{j\in J(n)} v_j x_j \text{ subject to} \\
l_{kS} \le \sum_{j\in S} x_j \le u_{kS} \text{ for } S\in\mathfrak{T}_k,\ k=1,\dots,K \\
l_{0S} \le \sum_{j\in S} \rho_j x_j \le u_{0S} \text{ for } S\in\mathfrak{T}_0 \\
\textstyle\sum_{\{j\in J(n)|k_j=k\}} x_j = q_k \text{ for } k=1,\dots,K
\end{aligned}
\tag{3}
$$

Because the vector $(q,x)\equiv 0$ satisfies all the constraints in (3), the zero bundle $q=0$ is feasible. By a theorem of linear programming, the set of vectors q for which the problem is feasible is a closed, bounded, convex set $Q\subseteq\Re^K$ and V is a continuous, concave function on that set. The next step is to define assignment messages and certain related concepts.

Definitions.

1. The *demand correspondence* for V is $D(p) = \arg\max_{q\in Q} V(q) - p\cdot q$.

2. The *indirect profit function* for V is $\pi(p)\equiv \max_{q\in Q} V(q) - p\cdot q$.

3. The valuation V is *substitutable* if for all prices $p, p'\in\Re^K_+$ and all $k=1,\dots,K$, if $D(p)=\{x\}$ and $D(p_{-k}, p'_k)=\{x'\}$ are singletons and $p'_k > p_k$, then $x'_{-k}\ge x_{-k}$.

4. A collection of sets $\mathfrak{T}$ is a *tree* if (1) for any two non-disjoint sets $S,S'\in\mathfrak{T}$, either $S\subset S'$ or $S'\subset S$ and (2) $\mathfrak{T}$ contains a largest set – the union of all its elements. That largest set is the *root* of $\mathfrak{T}$.

5. Given a tree of sets $\mathfrak{T}$, its *extended predecessor function* (P) maps each element of $\mathfrak{T}$, excluding the root R, into its unique predecessor and maps each $j \in R$ into the smallest set S satisfying $j \in S \in \mathfrak{T}$.

6. A *bound forest* is a collection of trees and associated bounds $\{\mathfrak{T}_0, ..., \mathfrak{T}_K, \{(l_{kS}, u_{kS}) \mid S \in \mathfrak{T}_k, k = 0, ..., K\}\}$ with all $l_{kS} \le 0 \le u_{kS}$. The trees satisfy:

   a. The root of $\mathfrak{T}_0$ is $R_0 = J(n)$ and, for $k = 1, ..., K$, the root of $\mathfrak{T}_k$ is $R_k = \left\{ j \in J(n) \mid k_j = k \right\}$.

   b. For $k = 1, ..., K$, the terminal nodes of tree $\mathfrak{T}_k$ are the singleton sets $\{j\}$ with $j \in J(n)$ and $k_j = k$.

   c. All bounds except the root bounds are finite, $0 \ge l_S > -\infty$ and $0 \le u_S < +\infty$, but the bounds on the roots may be infinite, $0 \ge l_{R_k} \ge -\infty$ and $0 \le u_{R_k} \le +\infty$.

   d. For any singleton set $\{j\} \in \mathcal{T}_{k_j}$, $l_{k_j\{j\}} = l_j$.

7. An *assignment message* consists of a collection of bids $(k_j, v_j, \rho_j, l_j, u_j)$ and a bound forest $\{\mathfrak{T}_0, ..., \mathfrak{T}_K, \{(l_{kS}, u_{kS}) \mid S \in \mathfrak{T}_k, k = 0, ..., K\}\}$.

8. A *basic assignment message* is an assignment message with each $\rho_j = 1$ and with all bounds l_{kS} and u_{kS} integers.

9. An *assignment exchange* is a mechanism mapping profiles of assignment messages for each bidder *n* to an outcome pair $(q_1^*,...,q_N^*,p^*)$ where $q^* \in \arg\max_{\{q|q_n \in Q_n\}} \sum_{n=1}^N V_n(q_n)$ subject to $\sum_{n=1}^N q_{nk} = 0$ for $k = 1,...,K$ and p^* is a supporting price vector, that is, for $n = 1,...,N$, $q_n^* \in \arg\max_{q \in Q_n} V_n(q) - p^* \cdot q$ (equivalently, $p^* \in \arg\min_p \pi_n(p) + p \cdot q_n^*$).

10. A *basic assignment exchange* is an assignment exchange in which the messages are restricted to be basic assignment messages.

The basic assignment messages form an extension of the set of messages allowed by the Shapley-Shubik mechanism. In the Shapley-Shubik mechanism, each participant occupies just one role, as a buyer or a seller. Each seller message includes just one bid ($|J(n)|=1$) and each buyer message includes just one bid for each product. If participant *n* is a seller, then the constraints on its one bid are $l_1 = -1$ and $u_1 = 0$. If participant *n* is a buyer, then its constraint bounds for each bid are $l_j = 0$ and $u_j = 1$ and its one group constraint has bounds $l_{R_{n0}} = 0$ and $u_{R_{n0}} = 1$. The basic assignment message space extends this Shapley-Shubik message space by allowing more bids, more constraints, and general integer bounds.

The three main results of this section can now be stated. Proofs follow just below.

<u>Theorem 1</u>. If participant *n* reports an assignment message, then its valuation $V : q \to \Re$ as given by (3) is continuous, concave and substitutable and its indirect profit function is submodular.

Theorem 2. If every participant *n* reports a continuous, concave substitutable valuation on a convex, compact set Q_n, then the set of market-clearing prices for the report profile is $\arg\min_p \sum_{n=1}^{N} \pi_n(p)$. This set is a non-empty, closed, convex sublattice.

Theorem 3. If every participant reports a basic assignment message, then there is an integer vector $q^* \in \arg\max_{q\in\Re^{NK}} \sum_{n=1}^{N} V_n(q_n)$ subject to $\sum_n q_{nk} = 0$ for all *k*.

The proof of theorem 1 makes use of the following results, which are also of independent interest.

Lemma 1. Suppose that the valuation function *V* is such that the corresponding indirect profit function π is well defined. Then *V* is substitutable if and only if its indirect profit function π is submodular.[9]

Lemma 2. Suppose $\pi(p) = \min_z g(z)$ subject to $(z, p) \in S$, where *g* is submodular, *S* is a sublattice in the product order, and *p* is a parameter. Then, π is submodular.

Proof of Lemma 1. Since π is convex on $\Re^K$, it is locally Lipschitz and differentiable almost everywhere. By Hotelling's lemma, the demand set is a singleton $D(p) = \{x(p)\}$ at exactly those points of differentiability and $\pi_k(p) \equiv \partial\pi(p)/\partial p_k = -x_k(p)$. Substitutability is equivalent to the condition that for $k = 1,\ldots,K$, $x_{-k}(p)$ non-decreasing in p_k. Submodularity is equivalent to the condition that on the same domain, $\pi_k(p)$ is non-increasing in $p_{k'}$ for $k' \neq k$. **QED**

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[9] Earlier versions of this result, as in Ausubel and Milgrom (2002) or Milgrom and Strulovici (2008), impose additional restrictions, such as discreteness of the goods, which are appropriate for those contexts. This version drops the unnecessary additional assumptions.

Proof of Lemma 2. Let p and p' be two price vectors and let z and z' be corresponding optimal solutions, so that $\pi(p) = g(z)$, $\pi(p') = g(z')$, and $(z, p), (z', p') \in S$. Since S is a sublattice, $(z \wedge z', p \wedge p'), (z \vee z', p \vee p') \in S$. By the definition of π, $\pi(p \wedge p') \le g(z \wedge z')$ and $\pi(p \vee p') \le g(z \vee z')$. Since g is submodular, $g(z \wedge z') + g(z \vee z') \le g(z) + g(z')$. Hence, $\pi(p \wedge p') + \pi(p \vee p') \le \pi(p) + \pi(p')$. **QED**

Proof of Theorem 1.[10] Let P_k denote the extended predecessor function associated with tree $\mathfrak{T}_k$. Let $\rho_{\{j\}} = \rho_j$ and $\rho_S = 1$ if $|S| \ne 1$. Using the tree structures, for $k = 0, \ldots, K$ and $S \in \mathfrak{T}_k$, one can define variables as follows:[11]

$$x_{kS} = \begin{cases} \sum_{S' \in P_0^{-1}(S)} \rho_{S'} x_{S'} & \text{for } k = 0 \\ \sum_{S' \in P_k^{-1}(S)} x_{S'} & \text{for } k = 1, \ldots, K \end{cases}$$

Using this augmented vector x, and with the notational convention that $x_{0j} \equiv x_{k_j j} \equiv x_j$, rewrite (3) as:

$$\begin{aligned} V(q) = \max_x \sum\nolimits_{j \in J(n)} & v_j x_j \text{ subject to} \\ & -x_{kS} + \sum\nolimits_{S' \in P_k^{-1}(S)} x_{kS'} = 0 \text{ for } S \in \mathfrak{T}_k, k = 1, \ldots, K \\ & x_{0S} - \sum\nolimits_{S' \in P_0^{-1}(S)} \rho_{S'} x_{0S'} = 0 \text{ for } S \in \mathfrak{T}_0 \\ & l_{kS} \le x_{kS} \le u_{kS} \text{ for } S \in \mathfrak{T}_k, k = 0, \ldots, K \\ & x_{kR_k} = q_k \text{ for } k = 1, \ldots, K \end{aligned} \tag{4}$$

The indirect profit function is:

[10] Intuitively, the argument sets up lemma 2 by showing that high goods prices are associated with low shadow prices on buyer upper bound constraints and reversely for seller constraints.

[11] We adopt the convention that if a set is empty, the sum it indexes is zero.

$$
\begin{aligned}
\pi(p) &= \max_{q\in\Re^K} V(q) - p\cdot q \\
&= \max_x \sum_{j\in J(n)} v_j x_j - \sum_{k=1}^{K} p_k x_{kR_k} \text{ subject to} \\
& -x_{kS} + \sum_{S'\in P_k^{-1}(S)} x_{kS'} = 0 \text{ for } S\in\mathfrak{T}_k, k=1,\ldots,K \\
& x_{0S} - \sum_{S'\in P_0^{-1}(S)} \rho_{S'} x_{0S'} = 0 \text{ for } S\in\mathfrak{T}_0 \\
& l_{kS} \le x_{kS} \le u_{kS} \text{ for } S\in\mathfrak{T}_k, k=0,\ldots,K
\end{aligned} \tag{5}
$$

Applying the duality theorem of linear programming, with λ_S^u and λ_S^l the shadow prices on the upper and lower bound constraints and μ_{kS} the shadow prices on equality constraints:

$$
\begin{aligned}
\pi(p) = \min_{\lambda,\mu} & \sum_{k=0}^{K} \sum_{S\in\mathfrak{T}_k} \left(u_{kS}\lambda_{kS}^u - l_{kS}\lambda_{kS}^l\right) \\
& \text{subject to} \\
& -\rho_j \mu_{0P_0(j)} + \mu_{kP_k(j)} \ge v_j \text{ for } j\in R_k, k=1,\ldots,K \\
& \lambda_{kS}^u - \lambda_{kS}^l + \mu_{kP_k(S)} - \mu_{kS} \ge 0 \text{ for } k=1,\ldots,K, S\in\mathfrak{T}_k - \{R_k\} \\
& \lambda_{0S}^u - \lambda_{0S}^l - \rho_S \mu_{0P_0(S)} + \mu_{0S} \ge 0 \text{ for } S\in\mathfrak{T}_0 - \{R_0\} \\
& \lambda_{kR_k}^u - \lambda_{kR_k}^l - \mu_{kR_k} \ge -p_k \quad \text{for } k=1,\ldots,K \\
& \lambda_{0R_0}^u - \lambda_{0R_0}^l + \mu_{0R_0} \ge 0
\end{aligned} \tag{6}
$$

For $k=0,\ldots,K$ and $S\in\mathfrak{T}_k$, define functions $f_{kS}(z) \equiv u_{kS}\max(0,z) + l_{kS}\min(0,z)$. Notice that these functions f_{kS} are convex and that, because either $\lambda_{kS}^u = 0$ or $\lambda_{kS}^l = 0$ (the upper and lower bound constraints on x_{kS} cannot both be binding), $f_{kS}(\lambda_{kS}^u - \lambda_{kS}^l) =$ $u_{kS}\lambda_{kS}^u - l_{kS}\lambda_{kS}^l$. Define $\theta_{kS} = \mu_{kS} - \left(\lambda_{kS}^u - \lambda_{kS}^l\right)$ for $k=1,\ldots,K$ and $\theta_{0S} = \mu_{0S} + \left(\lambda_{0S}^u - \lambda_{0S}^l\right)$. Substituting into (6) results in the following:

$$\pi(p)=\min_{\lambda,\mu}\sum_{k=1}^{K}\sum_{S\in\mathfrak{T}_k} f_{kS}\left(\mu_{kS}-\theta_{kS}\right)+\sum_{S\in\mathfrak{T}_0} f_{0S}(\theta_{0S}-\mu_{0S})$$

$$\begin{aligned}
&\text{subject to}\\
&-\rho_j\mu_{0P_0(j)}+\mu_{kP_k(j)}\geq v_j \text{ for } j\in R_k, k=1,\ldots,K\\
&\mu_{kP_k(S)}-\theta_{kS}\geq 0 \text{ for } k=1,\ldots,K, S\in\mathfrak{T}_k-\{R_k\}\\
&-\rho_S\mu_{0P_0(S)}+\theta_{0S}\geq 0 \text{ for } S\in\mathfrak{T}_0-\{R_0\}\\
&-\theta_{kR_k}\geq -p_k \quad \text{for } k=1,\ldots,K\\
&\theta_{0R_0}\geq 0
\end{aligned} \tag{7}$$

Let $C=\sum_{k=0}^{K}\left|\mathfrak{T}_k\right|$ be the total number of constraints included in bidder *n*'s assignment message. Let $(\theta,\mu,p)\in\Re^{2C+K}$ be a vector listing the dual variables and prices. Using the usual product order, treat $\Re^{2C+K}$ as a lattice. Since the f_{kS} functions are convex, the objective in problem (7) consists of a sum of submodular functions of (θ,μ). Since the objective is a sum of submodular functions of (θ,μ), it, too, is a submodular function of (θ,μ). Also, by inspection, each constraint in (7) defines a sublattice of $\Re^2$ for some one or two variables among (θ,μ,p) and hence of the higher dimensional space of vectors (θ,μ,p). Since an intersection of sublattices is a sublattice, the constraints in (7) define a sublattice.

Thus, (7) takes the form $\min_z g(z)$ subject to $(z,p)\in S$ where *g* is submodular and *S* is a sublattice. Lemma 2 applies, so π is submodular. Lemma 1 then applies, so *V* is substitutable. **QED**

<u>Proof of Theorem 2</u>. Since the corresponding primal problem can be represented as a continuous concave maximization on a compact set, the maximum exists and coincides with the minimum of the dual. Since the valuations are concave, the set of

market-clearing prices is the set of solutions to the dual problem: $\arg\min_p \sum_{n=1}^{N} \pi_n(p)$.

Since each π_n is continuous and convex, the set of minimizers of the dual problem is closed and convex. Since each π_n is submodular, by a theorem of Topkis (1978), the set of minimizers of the dual problem is a sublattice. **QED**

Proof of Theorem 3. We show something stronger than claimed by the theorem, namely, that there is an integer solution x^* to the problem that determines the goods assignments:

$$\begin{aligned}
&\max_x \sum\nolimits_n \sum\nolimits_{j\in J(n)} v_j x_j \text{ subject to} \\
&-x_{nkS} + \sum\nolimits_{S'\in P_{nk}^{-1}(S)} x_{nkS'} = 0 \text{ for } S \in \mathfrak{T}_{nk}, k=1,\ldots,K, n=1,\ldots,N \\
&x_{n0S} - \sum\nolimits_{S'\in P_{n0}^{-1}(S)} x_{n0S'} = 0 \text{ for } S \in \mathfrak{T}_{n0}, n=1,\ldots,N \\
&l_{nkS} \le x_{nkS} \le u_{nkS} \text{ for } S \in \mathfrak{T}_{nk}, k=0,\ldots,K, n=1,\ldots,N \\
&\sum\nolimits_n x_{nkR_{nk}} = 0 \text{ for } k=1,\ldots,K
\end{aligned} \tag{8}$$

The sign restrictions $l_{nkS} \le 0$ and $u_{nkS} \ge 0$ ensure that $x \equiv 0$ satisfies the constraints of the problem, so the problem is feasible. The individual bounds on each x_j imply that the constraint simplex is bounded. For a feasible, bounded linear program, there is always an optimal solution at a vertex of the constraint simplex. Hence, to prove the theorem, it is sufficient to show that every vertex of the simplex defined by the constraints in (8) is an integer vector.

Each vertex of the constraint simplex is determined by a set of binding upper and lower bound constraints of the form $x_S = u_S$ or $x_S = l_S$ and the equation $\mathbf{Ax} = \mathbf{0}$, which describes the equality constraints in (8). Fix any vertex and denote the right-hand sides of the binding upper and lower bound constraints by $\bar{\mathbf{u}}$ and $\bar{\mathbf{l}}$, which by hypothesis are integer vectors. Write the vector $\mathbf{x}$ in the form $(\hat{\mathbf{x}}, \bar{\mathbf{x}}_l, \bar{\mathbf{x}}_u)$ where the binding constraints are $\bar{\mathbf{x}}_l = \bar{\mathbf{l}}$, $\bar{\mathbf{x}}_u = \bar{\mathbf{u}}$, which we write as $\bar{\mathbf{x}} = \bar{\mathbf{b}} = (\bar{\mathbf{u}}, \bar{\mathbf{l}})$. Let $\bar{\mathbf{A}}$ and $\hat{\mathbf{A}}$ be the matrices consisting of the columns of **A** corresponding to $\bar{\mathbf{x}}$ and $\hat{\mathbf{x}}$, respectively. Then the equation $\mathbf{Ax} = \mathbf{0}$ is the same as $\mathbf{0} = \mathbf{Ax} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \bar{\mathbf{A}}\bar{\mathbf{x}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \bar{\mathbf{A}}\bar{\mathbf{b}}$. Taking $\mathbf{b} \equiv -\bar{\mathbf{A}}\bar{\mathbf{b}}$, we have $\hat{\mathbf{A}}\hat{\mathbf{x}} = \mathbf{b}$, where **b** is an integer vector.

It is therefore sufficient to show that for every non-singular submatrix $\hat{\mathbf{A}}$ of **A** and every integer vector **b**, there is an integer solution $\hat{\mathbf{x}}$ to $\hat{\mathbf{A}}\hat{\mathbf{x}} = \mathbf{b}$. For this, it suffices to show that **A** is *totally unimodular*.[12] According to a theorem of Hoffman (see Heller and Tomkins (1956)), a matrix is totally unimodular if all the entries of **A** are elements of the set $\{0, +1, -1\}$, if each column of **A** has at most two non-zero entries, and if no two non-zero entries in any column have the same sign. We finish by verifying the Hoffman conditions.

Examine the columns of **A** as represented in (8) which correspond to the variables x_{nkS}. For $k = 0$ and $S=R_{n0}$ the root of a $\mathfrak{T}_{n0}$ tree for some participant *n*, x_{n0S} appears in only one equality constraint in (8) and so has a single entry in its column. For $k = 1, \ldots, K$, each of the variables $x_{nkR_{nk}}$ appears twice: once in its defining equation and again in the market-clearing constraint for *k*, and its two coefficients have opposite signs. For $k = 1, \ldots, K$ and all sets $S \in \mathfrak{T}_{nk} - \{R_{nk}\}$, x_{nkS} appears twice: once with coefficient -1 in the equation defining x_{nkS} and once with coefficient +1 in the equation defining $x_{nkP_{nk}(S)}$. For $k = 0$ and $S \in \mathfrak{T}_{n0} - \{R_{n0}\}$, x_{n0S} appears twice: once with coefficient +1 in its defining

[12] See the *Wikipedia* entry on "unimodularity" for an accessible treatment of the relevant mathematics.

equation and once with coefficient –1 in the equation defining $x_{n0P_{n0}(S)}$. Last are the x_j variables. Recall that by our extended definition of predecessor, $j \in P^{-1}{}_{nk}(S)$ for exactly two sets, one in $\mathfrak{T}_{nk_j}$ with coefficient +1 and one in $\mathfrak{T}_{n0}$, with coefficient –1. Hence, the Hoffman conditions are satisfied. **QED**

III. Partial Converse to Theorems 1 and 2

The structure of assignment messages allows bidders to report values and effectiveness coefficients without limitations but restricts the form of constraints to be a bound forest. This section shows that if one fails to impose the constraint that $\mathfrak{T}_{n0}$ is a tree, then theorems 1 and 2 become invalid.

The main idea can be illustrated with the example of a buyer for whom the lower bounds l_j and l_{kS} are all zero. Suppose that there are three goods and that this buyer has three bids, $j = 1, 2, 3$, each with $v_j = 2.9$ and $k_j = j$, all constrained so that $0 \le x_j \le 2$. Suppose that the multi-product groups constraints in the problem are $x_1 + x_2 \le 3$ and $x_2 + x_3 \le 3$, violating the tree structure. Then, for the price vector (0,1,2), the corresponding demand is (2,1,2) and for the price vector (3,1,2), the corresponding demand is (0,2,1): raising the price of good 1 reduces the demand for good 3, violating the substitutes condition. Moreover, if the available quantities are one unit of good 2 and two units each of goods 1 and 3, then the market clears for price vectors (0,1,2) or (2,1,0) but not for the join, which is (2,1,2), so the set of clearing prices in this example is not a sublattice.

More generally, given *any* set of constraints $\mathfrak{T}_{n0}$ over goods that fails to have the tree structure, we can find a similar counter-example as follows. Since the constraints do not form a tree, there are two sets, $S, S' \in \mathfrak{T}_{n0}$ such that each of $S - S'$, $S \cap S'$, and $S' - S$ are non-empty. Let goods 1, 2 and 3 index goods that are denote elements of these sets and specify that the values of any other goods are zero. Let the bounds constraining these goods be given as in the preceding paragraph and let the bounds on all other constraints be very large, so that those constraints do not bind. These specification reproduce precisely the example of the preceding paragraph for any $\mathfrak{T}_{n0}$ that is not a tree. That proves the following theorem.

Theorem 4. If the set $\mathfrak{T}_{n0}$ is not a tree, then there exist bids and integer bounds for each $S \in \mathfrak{T}_{n0}$ and supplies for the other participants such that the valuation V_n is not a substitutes valuation, the indirect profit function π_n is not submodular, and the set of market-clearing prices is not a sublattice.

IV. Tightness

A *direct mechanism* is a triple (N, M, ω), where *N* is the set of participants, *M* is the product space of types (“message profiles”), and $\omega : M \to \Omega$, where Ω is the set of possible outcomes. The mechanism $(N, \hat{M}, \omega)$, where $\hat{M}_n \subseteq M_n$, is a *simplification* of the mechanism (N, M, ω). For tightness analysis, it is assumed that $\Omega = \times_{n \in N} \Omega_n$, where each Ω_n is a topological space, and that each player *n*’s payoff is by $u_n(\omega_n)$, where the payoff function u_n is continuous.

A simplified direct mechanism has the *outcome closure property* if for every player *n*, strategy profile $\hat{m}_{-n} \in \hat{M}_{-n}$ and strategy $m_n \in M_n$, and every open set *O* such that $\omega_n(m_n, \hat{m}_{-n}) \in O$, there is a strategy $\hat{m}_n \in \hat{M}_n$ such that $\omega_n(\hat{m}) \in O$. In words, this means that when other participants are limited to using messages their restricted message space, limiting *n* to do the same has little or no effect on the set of outcomes that *n* can bring about. The mechanism $(N, \hat{M}, \omega)$ is a *tight* simplification of (N, M, ω) if for all utility profiles $u = (u_n)_{n \in N}$ and every $\varepsilon \geq 0$, every pure-strategy profile that is an ε-Nash equilibrium of the simplified mechanism is also an ε-Nash equilibrium of the original, extended mechanism. The Simplification Theorem of Milgrom (2008) asserts that if $(N, \hat{M}, \omega)$ has the outcome closure property with respect to (N, M, ω), then the simplification is tight.

For this application, we take $\omega_n = (q_n, p)$, which permits each participant to care about his goods assignment and about the prices, but not the goods assigned to others. In standard equilibrium theory, preferences for a participant *n* depend only on $(q_n, p \bullet q_n)$ – his goods assignment and payment. By including the price vector in a more general way, the tightness analysis allows that a participant may prefer that its competitor's product commands a low price or that its partner's product commands a high price. It allows that a participant's actual preferences can be any that is acceptable for the Arrow-Debreu theory, but it is not limited to those preferences and certainly not limited to the preferences that are describable using assignment messages.

The next theorem applies not just to the general assignment exchange, but also to mechanisms that limit the messages participants can use to a subset of the assignment messages. To describe the permissible limitations on messages, let us say that an assignment message m_n is *minimally constrained* if its only finite constraint bounds (l_S, u_S) correspond to the singleton sets $S = \{j\}$. An *elementary assignment message m_n* for participant *n* is an assignment message that is minimally constrained and includes at most two bids for any product *k*: $|\{j \in J(n) : k_j = k\}| \leq 2$ for $k = 1,...,K$.

Theorem 5. Any simplified Walrasian exchange in which each bidder *n*'s message space contains *only* assignment messages but contains *all* elementary assignment messages satisfies the outcome closure property with respect to the full Walrasian exchange and (hence) is a tight simplification.

Proof. Let $\hat{M}_n$ be bidder *n*'s simplified message space and let M_n be the full Walrasian message space. Fix a participant *n* and messages $\hat{m}_{-n} \in \hat{M}_{-n}$ and $m_n \in M_n$. Let $(p,q) \equiv \omega(\hat{m}_{-n}, m_n)$. Let $\sigma_{nk} = sign(q_{nk}) \in \{-1,0,1\}$ and fix $\varepsilon > 0$. Since *n*'s message space includes all elementary assignment messages, it includes the message $\hat{m}_n$ with bids $j = 1,...,2K$ as follows. For $k = 1,...,K$, $k_{2k-1} = k$, $v_{2k-1} = p_k + \sigma_{nk}\varepsilon$, $v_{2k} = p_k - \sigma_{nk}\varepsilon$, $u_{2k-1} = u_{2k} = \max(0, q_{nk})$ and $l_{2k-1} = l_{2k} = \min(0, q_{nk})$. The message $\hat{m}_n$ specifies no other finite bounds. Let $(\hat{p}, \hat{q})$ be the competitive equilibrium outcome selected by the mechanism when the message profile is $\hat{m}$.

Since (p,q) is a competitive equilibrium for the report profile $(\hat{m}_{-n}, m_n)$ q_n solves $\max_{x_n} \max_{\{x_{-n} | \sum_{l \neq n} x_n = 0\}} \left(V_n(x_n | m_n) + \sum_{l \neq n} V_l(x_l | \hat{m}_l) \right)$. And since *n* demands q_n at prices *p*, (p,q) is also a competitive equilibrium for report profile $\hat{m}$. From that and the fact that $\varepsilon > 0$, q_n *uniquely* solves $\max_{x_n} \max_{\{x_{-n} | \sum_{l \neq n} x_n = 0\}} \left(V_n(x_n \mid \hat{m}_n) + \sum_{l \neq n} V_l(x_l \mid \hat{m}_l) \right)$.

Hence, even though there may be multiple competitive equilibria for the message profile $\hat{m}$, all assign the bundle q_n to participant *n*: $\hat{q}_n = q_n$. Moreover, since every market-clearing price vector support this choice by *n*, the price vector $\hat{p}$ must satisfy $p_k - \varepsilon \leq \hat{p}_k \leq p_k + \varepsilon$ for every product *k*. Since ε can be arbitrarily small, the outcome closure property is proved. Tightness then follows from Milgrom's Simplification Theorem. **QED**

V. Additional Connections

One connection is between the assignment exchange and single product exchanges. If $K = 1$, the assignment exchange reduces to what the literature calls a *double-auction*. Each participant's report describes a step-function supply or demand curve and these are intersected to determine market-clearing prices and quantities. In case the market-clearing prices or quantities are not unique, any selection rule is consistent with the assignment exchange.[13]

Another connection is to the Vickrey auction. In such an auction, if a participant *n* acquires a single good *k*, it pays the opportunity cost of that good, which is equal to the incremental value of one additional unit of good *k* to the coalition of *other* participants. In the linear program for the basic assignment exchange, the lowest market-clearing price p_k for good *k* is its shadow price – the amount by which the optimal value would

[13] In one-sided cases (with just bids to buy and a fixed supply, or bids to sell and a fixed demand), the kinds of problems found in share auctions (Wilson (1979)) can present themselves. Typical solutions to these problems, such as proposed in McAdams (2002) and Kremer and Nyborg (2004), can be adapted to the assignment exchange.

increase if an additional unit of good *k* were made available to the coalition of *all* players. If participant *n* has demand for just one unit in total and acquires a unit of good *k*, then the additional unit for the coalition of all participants is actually assigned to someone besides *n*, so p_k is the increased optimal value of that unit to the other participants – *n*'s Vickrey price.

Theorem 6. Suppose that some participant *n* bids to acquire at most one unit in a basic assignment exchange and that the exchange selects the price vector *p* that is the minimum market-clearing price vector. Then, if *n* acquires a unit of good *k*, the price p_k is equal to *n*'s Vickrey price for *k*.

A symmetric statement can be made about participants who *sell* one unit and exchanges that select the *maximum* market-clearing price vector.

VI. From Theory to Practice

As described in the introduction, the implementation of multi-product clock auctions can be handicapped by finite bid increments, scheduling issues, and short market periods. The assignment exchange avoids these problems.

The two main practical limitations of assignment exchanges are associated with their enforced simplification of preference messages and the paucity of information they reveal to bidders during the auction. The latter may be significant when there is some common value element in the environment or when a bidder's payoff depends on the trades made by other bidders.

To assess the scope and limits of the assignment message space, it is helpful to begin by studying cases in which simple basic assignment messages can be effective.

Suppose, for example, that an electricity buyer *n* can purchase power from any of three sources, 1, 2 or 3, subject to transmission costs (t_1, t_2, t_3) and transmission capacity limits (U_1, U_2, U_3). If *n* needs to buy *P* units of power and the value per unit is α, then bids $j = 1, 2, 3$ with $k_j = j$, $v_j = \alpha - t_j$, $u_j = U_j$, $l_j = 0$ and one constraint for $S = \{1, 2, 3\}$ with $u_S = P$ and $l_S = 0$ accurately expresses the bidder's demand. If there are also transmission losses to account for, the bidder can handle those by setting $\rho_j < 1$; otherwise, $\rho_1 = \rho_2 = \rho_3 = 1$.

In the same setting, it might happen that the buyer has already acquired all of its power need for some time period but would be willing to sell up to β units power at A in exchange for β units at B or C, provided the price is right. This swap can be encoded with three bids and the constraints: $0 \geq x_A \geq -\beta$, $x_B, x_C \geq 0$, $x_A + x_B + x_C = 0$ (which is encompassed by the theory because it can be expressed using upper and lower bounds: $0 \leq x_A + x_B + x_C \leq 0$).

Swap bids have the potential to add liquidity to an exchange hindered by lack of volume. Investigating this fully is beyond the scope of this paper: it requires a theory of why owners do not constantly participate in and provide liquidity to markets. Nevertheless, it is clear that in a market with modest liquidity, swaps encourage participation by limiting the risk that one part of an intended transaction might be executed without the other parts. For example, with separate markets, a swapper with a budget limit might have to sell one commodity before buying the other in order to raise funds to transact, leaving the swapper exposed to the risk of not finding a seller for the other part of the planned transaction. By eliminating such risks, swaps make participation safer, increasing liquidity.

The power of simple assignment messages in the examples given above is important because simplicity is often a design goal. One might simplify the general assignment exchange by limiting the number of bids, constraints, or levels in the constraint trees. Theorems 1, 2, 3 and 5 have been constructed to apply even to exchanges that incorporate additional simplifications.

One kind of common constraint that is not fully reflected in theorem 5 arises when the exchange limits a participant's role. For example, only certain parties may be qualified sellers of particular goods, as implemented by a restriction limiting when $l_j < 0$ is permitted. This can be significant for conclusions about tightness, and it is natural to investigate extensions of theorem 5 by imposing similar restrictions on the related Walrasian exchange. I leave that task for others.

Another common limitation imposed by operators is a credit limit on buyers. Whether this is implemented as a limit on the maximum acceptable bid from a bidder or as a limit on the maximum quantities that can be demanded, the result is simply to restrict the bidder to a subset of the assignment message space, so the theorems continue to apply.

When bidder market power in an auction is alleged, it may be good policy to limit the total quantity of all goods or only of certain goods *k* purchased by some set of bidders. Such a policy leads to constraints that are complex because they combine bids across bidders. One approach is by product redefinition. For example, if the operator wants to limit bidders 1 and 2 to purchase no more than half of the available units of good 1, it can accomplish that by splitting good 1 into types 1A and 1B and restricting bidders 1 and 2 from bidding on type 1B. This procedure has precedent: it is similar to the set-asides used by the US Federal Communications Commission to restrict purchases by incumbents in some auctions.[14]

Whether the assignment messages are sufficiently encompassing is likely to vary by application. Certainly, scale economies and complements among lots are sometimes important and cannot generally be solved merely by redefining lots. For example, in electricity, generating plants typically have large fixed costs that require all or nothing decisions about whether to use their power capacity. While such limits are not directly expressed using assignment messages, it is often possible to use the assignment exchange as part of a solution. One *ad hoc* procedure is to operate the exchange in two or more rounds to allow preliminary price discovery to guide bids at the final round. This does not entirely eliminate the fixed cost problem, but it may sometimes mitigate it. Staged dynamics of this sort may also be helpful when there are important common value elements or when bidders can invest in information gathering during the process, as in Compte and Jehiel (2000) or Rezende (2005).

A more exact procedure incorporates the assignment exchange as an element within a general combinatorial auction or exchange. For example, participants might be allowed to report fixed costs of transacting in addition to their assignment messages. Doing that would lead to a two-stage problem, in which finding the right set of participants is a combinatorial optimization problem, but finding the allocation for a

[14] The FCC combined this with restrictions on post-auction transfers to limit gaming of the system.

given set of participants is an assignment exchange problem. Similarly, in the airline slot problem, if there is no single time *T* that is covered by all the relevant intervals, it may still be possible to organize the optimization around a limited number of such times – the combinatorial part of the problem – and to allow the assignment exchange to solve the remaining part.

Three key properties of assignment and basic assignment messages – that they are simple to use and express only substitutable preferences and that basic assignment messages lead to efficient integer solutions – make them potentially valuable for simplifications of other mechanisms in addition to the Walrasian exchange. For example, two principal disadvantages of Vickrey auctions – complexity of the message space and "low" seller revenues (less than in any core allocation) – hinge on either the complexity of the message space and the availability of messages that report non-substitutable values, respectively.[15] A simplified Vickrey auction in which bidders are limited to reporting assignment messages escapes these disadvantages. As another example, consider assignment problems with discrete goods and rules against cash transfers, such as the problems of assigning students to courses or flight attendants to routes. In such cases and assuming that basic assignment messages describe ordinal preferences, by maximizing welfare-weighted sums of assignment values using linear programming, one identifies all and only integer efficient solutions. And, if budget constraints are imposed to find competitive equilibrium solutions, the resulting fractional allocations can be shown to correspond to a randomization over integer solutions.[16]

[15] Milgrom (2004), sections 2.5 and 8.1, and Ausubel and Milgrom (2006).
[16] See Budish, Che, Kojima, and Milgrom (2008).

As described in the introduction, direct, sealed-bid mechanisms enjoy important advantages compared to ascending or descending auctions, particularly for time-sensitive applications. Assignment exchanges, in particular, are tight, simple to use, fast to execute, and precise in determining both equilibrium prices and goods assignments. Assignment messages provide a compact expression of a useful set of substitutable preferences for a range of applications and the basic assignment messages ensure that equilibrium assignments entail only integer quantities. The assignment exchange design is *robust*, in the sense that its key properties remain even when the assignment message space is further restricted in any way that does not eliminate any *simple* assignment messages, and *maximal* in the sense no extension of the bid tree constraint architecture is possible without destroying the key substitutes property of the message space. In combination, these attributes make the assignment exchange an attractive candidate for the many practical applications in which the goods or items to be assigned are substitutes.

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