Results on efficiency in school choice

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Abstract

The deferred acceptance (DA) algorithm is used in important real-world markets like school choice (the matching of students to schools), and the medical assignment process (assignment doctors to hospitals). However, one drawback of DA is that the matching it produces is not always Pareto efficient for the proposing side. This is problematic because (1) we do not know in which markets DA will produce Pareto efficient matchings and (2) in markets when we use DA, there is space for efficiency improvement. In this paper, we propose a mechanism that improves the efficiency of DA and is asymptotically strategy-proof. We further show that, for sufficiently random markets satisfying the condition that the number of students does not exceed the number of school seats, DA is asymptotically efficient. This helps policy by (1) allowing for improvement in efficiency in those markets where DA is already used and (2) further informing in which markets DA can be used efficiently.

1. Introduction

A number of school systems across the United States, including New York and Boston public schools, have adopted the student-proposing deferred acceptance mechanism (DA) to assign students to schools. The deferred acceptance mechanism works as follows. Given a finite set of students $S$, a finite set of schools $C$, the preferences of each student, and the priority structure and capacity of each school, the deferred acceptance mechanism implements the following algorithm: at step $1$, each student applies to her most preferred school, and each school tentatively accepts their most preferred students up to capacity, rejecting all other applicants to the school. At step $t$, each rejected student applies to her most preferred school that has not yet rejected her. Each school tentatively accepts their most preferred students up to capacity, and rejects all other applicants to the school. The algorithm terminates once no rejections occur or once all rejected students have no further schools to apply to. Because the set of schools and students is finite, termination occurs in a finite number of steps, and the tentative matching at the last step becomes the finalized matching for students and schools.

To illustrate how the deferred acceptance mechanism works, consider the following example.

Example 1. Suppose we have a set $S = \{s_1, s_2, s_3\}$ of students and a set $C = \{c_1, c_2, c_3\}$ of schools. Suppose each school has capacity one and student preferences are given by:

$\succ s_1: c_1, c_2, c_3$

$\succ s_2: c_1, c_3, c_2$

$\succ s_3: c_3, c_1, c_2$

where the notational convention is that schools are listed in order of preference. For example, student $s_1$ has first choice school $c_1$, second choice school $c_2$, and third choice school $c_3$. School priorities are given by:

$\succeq_{c_1}: s_3, s_1, s_2$

$\succeq_{c_2}: s_1, s_2, s_3$

$\succeq_{c_3}: s_2, s_3, s_1$

where the notational convention is that students are listed in order of priorities; at school $c_1$, for instance, student $s_3$ has the highest priority, $s_1$ has the second highest priority, and student $s_2$ has the lowest priority.

The deferred acceptance algorithm would execute as follows:

- In step 1, students $s_1$ and $s_2$ apply to $c_1$, and student $s_3$ applies to $c_3$. School $c_1$ tentatively accepts student $s_1$ and rejects student $s_2$. School $c_3$ tentatively accepts student $s_3$. 
In step 2, student \( s_2 \) applies to her second choice school, \( c_3 \). School \( c_3 \) tentatively accepts student \( s_2 \) and rejects student \( s_3 \).

In step 3, student \( s_3 \) applies to \( c_1 \). School \( c_1 \) tentatively accepts student \( s_3 \) and rejects student \( s_1 \).

In step 4, student \( s_1 \) applies to school \( c_2 \). School \( c_2 \) tentatively accepts \( s_1 \). Since no rejections occur in this step, the algorithm terminates and the final matching is \( \mu = \{(s_1, c_2), (s_2, c_3), (s_3, c_1)\} \).

However, notice that all students would weakly prefer the matching \( \mu' = \{(s_1, c_2), (s_2, c_1), (s_3, c_3)\} \), with \( s_2 \) and \( s_3 \) strongly preferring this matching.

Roth (1982) shows that the student-proposing deferred acceptance algorithm is strategy-proof for students, that is, students never benefit by misreporting preferences. DA has a number of nice properties: it produces a stable matching, meaning that no student is matched with a school they consider unacceptable, and no student justifiably thinks that they should be at a more preferred school; when priority orders are strict, it produces the student-optimal stable matching, that is, the matching students most prefer of stable matchings; and it is, of course, strategy proof for students.

One drawback of DA is that it may not produce the best matching for students. In the above example, all students would weakly prefer the matching \( \mu' \) to the matching that DA produces. Intuitively, DA's efficiency loss in Example 1 occurs because it goes through a cycle: student \( s_1 \) kicks student \( s_2 \) out of school \( c_1 \), which causes \( s_2 \) to kick \( s_3 \) out of school \( c_2 \), which in turn causes \( s_3 \) to kick \( s_1 \) out of school \( c_1 \). Hence, all students are rejected in this rejection chain; and, all students would have been weakly better off if \( s_1 \) had never applied to \( c_1 \), because \( s_1 \) is ultimately rejected from \( c_1 \) anyway, and had \( s_1 \) not applied to \( c_1 \), the rejections of \( s_2 \) and \( s_3 \) could have been avoided. DA incurs this inefficiency loss to uphold stability: if DA had produced \( \mu' \), that is, if \( s_2 \) had been matched to \( c_1 \), \( s_1 \) would prefer \( c_1 \) to her match and \( c_1 \) would have preferred to take \( s_1 \) to the student assigned to \( c_1 \), meaning that \( s_1 \) would justifiably think that she should be at her more preferred school.

Example 2. When school priority orders are weak, students can suffer further efficiency loss through the introduction of artificial cycles via tie-breaking. Consider the following variant of Example 1, where \( c_1 \) is indifferent between \( s_1 \) and \( s_2 \):

\[
\succ_{c_1} : c_1, c_2, c_3 \\
\succ_{s_1} : c_1, c_3, c_2 \\
\succ_{s_2} : c_2, c_1, c_2 \\
\succeq_{c_1} : s_3, \{s_1, s_2\} \\
\succeq_{c_2} : s_1, s_2, s_3 \\
\succeq_{c_3} : s_1, s_3, s_1
\]

Where the notational convention is if students enclosed by a bracket at a particular school, those students have the same priority for that school; at \( c_1 \), for example, \( s_1 \) and \( s_2 \) have equal priority. Suppose we tiebreak between \( s_1 \) and \( s_2 \) at \( c_1 \) by having \( c_1 \) prioritize \( s_1 \) relative to \( s_2 \). Then the priority structure becomes as shown in Example 1 so DA produces matching \( \mu \). But all students would be better off under \( \mu' \), and the matching \( \mu' \) would be stable since \( c_1 \) is indifferent between \( s_1 \) and \( s_2 \). Hence, when school priority orders are weak, DA may not even produce a student-optimal stable match.

Research on how to improve the efficiency of DA is important because DA directly affects the real-life placements of students. In particular, a lack of efficiency means that at least one student could have been placed at a more preferred school without adversely affecting the placements of other students. DA's lack of efficiency likewise will affect real-world outcomes in any other market in which DA is used, for example, the National Residency Matching Program, matching doctors to residences.

Therefore DA's efficiency should factor into decisions of when and in which markets to use DA as the matching mechanism of choice. Furthermore, in those markets that already use DA, mechanisms that improve the efficiency of DA will in principal improve the assignment of some set of students.

Thus far, research on in what type of markets DA incurs efficiency loss is limited. Che and Terceux (2015) show that when student preferences are uncorrelated, in the sense that there is no common ranking of school tiers, DA will be efficient in large markets. Ehlers and Westkamp (2011) show that when school preferences...
are weak, DA will produce the student-optimal stable match (which, we recall may not be efficient) only when ties can be broken in a way that avoids introducing cycles into the priority structure.

Research on mechanisms to improve the efficiency of DA is more extensive. Stable improvement cycles (SIC), proposed by Erdil and Ergin, work by having students trade schools after their DA assignment. While SIC improves efficiency, it is not strategy proof. Kesten proposes removing cycles in DA by identifying student-school pairs that cause cycles, and removing those pairs from consideration in the DA process. This process again improves efficiency but is not strategy proof. When schools have weak preferences over students, Erdil (2014) and Eilfers and Westkamp (2011) provide conditions under which it is possible to improve on the efficiency of DA in a strategy-proof way. The primary barrier in implementing efficiency-improving mechanisms is these conditions do not seem to be usually met in school choice problems, and that the efficiency-improving mechanisms found so far have not been strategy proof in realistic environments.

This paper’s contribution is two-fold. First, we improve on Che and Terceux (2015) by providing conditions under which, even when students have common preferences over schools, DA will be efficient in large markets. In particular, we show that under sufficiently random preferences, so long as the number of schools a student may rank is fixed, DA becomes efficient as the number of colleges goes to infinity. This is because, with sufficiently random preferences, the probability that the same two colleges will be on two different students lists – a necessary condition for DA to incur inefficiency – goes to zero. Second, we provide an efficiency-improving mechanism that is similar to Kesten’s but whose efficiency-improvement is less than Kesten’s, and use the large market result, with more flexible conditions, to show that in large markets our efficiency-improving mechanism is approximately strategy-proof.

The importance of these results is, first, that they expand understanding of those markets in which DA can be used without efficiency concerns, and second, that they for the first time provide a somewhat realistic environment in which an efficiency-improving mechanism will be strategy proof. Presumably this efficiency-improving mechanism could therefore be implemented in such environments with limited concerns about strategy-proofness.

The paper proceeds as follows. Section 2 introduces the model generally used when discussing school choice. Section 3 provides an overview of the literature on improving efficiency in school choice. In particular, the overview will cover the following questions: Is DA ever efficient; in particular, is it efficient in large markets? What causes DA to incur efficiency loss? Can we improve the efficiency of DA ex-post? Can we improve the efficiency of DA ex-ante? Where does current research stand on the efficiency of DA, and is it desirable to extend such research? Section 4 present original results on efficiency, with motivation for each. Section 5 concludes.

2. Model

A market is a four-tuple $G = (S, C, (\succ_c)_{c \in SUC}, (q_c)_{c \in C})$. $S$ and $C$ are finite and disjoint sets of students and schools, respectively. For $s \in S$, $\succ_s$ is a preference relation over $C$ and being unmatched (being unmatched is denoted by $\emptyset$). We assume student preferences are strict. If $c \succ_s \emptyset$, we say that $c$ is acceptable to $s$. For each school $c$, $\succ_c$ is a priority order over the set of students. We allow priority orders to be weak. We write $s' \succ_c s$ if and only if $s' \succ_c s$ but not $s \succeq s'$. For each $c \in C$, $q_c$ is the total capacity of $c$. The set of priority orders is called a priority structure.

A matching $\mu$ is a mapping from $C \cup S$ to $C \cup S \cup \{\emptyset\}$ such that

1. $\mu(s) \in C \cup \{\emptyset\}$
2. For any $s \in S$ and $c \in C$, $\mu(s) = c$ if and only if $s \in \mu(c)$
3. $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$ for all $c \in C$

Given a matching $\mu$, the priority of student $s$ for school $c$ is violated if $c \succ_s \mu(s)$ and there exists a student $s_2$ such that $\mu(s_2) = c$ but $s \succ_c s_2$.

A matching $\mu$ is stable if

1. $\mu(s) \succeq_c \emptyset$ for each $s \in S$ and
2. No student’s priority for any school is violated.
3. No student would rather be matched with a school that has empty seats, that is, there is no student $s$ and school $c$ such that $c \succ_s \mu(s)$ and $|\mu(c)| < q_c$.
Given a matching \( \mu \), if there does not exist a matching \( \mu' \) such that \( \mu'(s) \succeq_s \mu(s) \) for all \( s \in S \) and \( \mu'(s) \succ_s \mu(s) \) for at least one \( s \in S \), we call \( \mu \) a student-optimal match, or equivalently, say that \( \mu \) is Pareto-efficient.

Given a stable matching \( \mu \), if there does not exist a stable matching \( \mu' \) such that \( \mu'(s) \succeq_s \mu(s) \) for all \( s \in S \) and \( \mu'(s) \succ_s \mu(s) \) for at least one \( s \in S \), then we will call \( \mu \) a student optimal stable match, or, equivalently, a constrained efficient matching.

2.1. Mechanisms

A mechanism is a function \( \phi \) that, for each market \( G \), associates a matching \( \phi(G) \). A mechanism \( \phi \) is stable if \( \phi(G) \) is a stable matching in \( G \) for any given \( G \).

A mechanism \( \phi \) is strategy proof if there is no way for a student to profitably misreport preferences, i.e. if there does not exist a market \( (S, C, \succeq = (\succeq_i)_{i \in S \cup C}, (q_e)_{e \in C}) \), a student \( s \) and preferences \( \succ_s \) such that \( \phi_s(S, C, \succeq_s, \succ_s, q_e) \neq \phi_s(S, C, \succeq, q_e) \). A mechanism \( \phi \) is constrained efficient if for any market \( G \) it always produces a constrained efficient matching for that market.

Given two mechanisms \( \phi \) and \( \gamma \), we will say that \( \phi \) dominates \( \gamma \) if for any given market \( G = (S, C, \succeq = (\succeq_i)_{i \in S \cup C}, (q_e)_{e \in C}) \), for all \( s \in S \), \( \phi(s) \succeq_s \gamma(s) \) and for at least one \( s \in S \), \( \phi(s) \succ_s \gamma(s) \).

We introduce a mechanism commonly used in school choice, the deferred acceptance algorithm (Gale and Shapley, 1962; adopted to centralized school choice by Abdulkadiroğlu and Sonmez, 2003):

- **Step 1**: Start with a matching in which no student is matched. Each student \( s \) applies to her first choice school (call it \( c \)). The school \( c \) rejects \( s \) if \( q_c \) seats are filled by students who have higher priority than \( s \) at \( c \). Each school \( c \) keeps all other students who applied to \( c \).

- **Step \( t \)**: Start with the tentative matching obtained at the end of Step \( t - 1 \). Each student \( s \) applies to her first choice school (call it \( c \)) among all schools that have not rejected \( s \) before. The school \( c \) rejects \( s \) if \( q_c \) seats are filled by students who have higher priority than \( s \) at \( c \). Each school \( c \) keeps all other students who applied to \( c \).

The algorithm terminates at a step in which no rejection occurs, and the tentative matching at that step is finalized. Because no student re-applies to a school that has rejected her, and by formulation of the algorithm at least one rejection occurs in the steps in which the algorithm does not terminate, the algorithm terminates in a finite number of steps. Modifying the argument by Gale and Shapley (1962), Abdulkadiroğlu and Sonmez (2003) show that, when priority orders are strict, the outcome of the deferred acceptance algorithm is the student-optimal stable matching, a stable matching that is unanimously most preferred by all students among all stable matchings.

Since standard DA requires the rankings to be strict on both sides, using DA requires a procedure to break ties if schools’ priorities are non-strict. We will denote a tie-breaking procedure by \( \tau \). Two common tie-breaking procedures are single tie-breaking (STB), which assigns every student a single lottery number uniform-randomly to break ties at every school, and multiple tie-breaking (MTB), which assigns a distinct lottery number to each student at every school.

When priorities are weak, the randomized deferred acceptance (RDA) mechanism uses STB or MTB to break ties, then runs DA.

2.1.1. Stochastic assignment mechanisms

A stochastic assignment is a profile \( \gamma = (\gamma_s)_{s \in S} \) of vectors \( \gamma_s = (\gamma_{sc})_{c \in C \cup \emptyset} \) such that \( \gamma_{sc} \in R_+ \) for all \( s \in S \) and \( c \in C \cup \emptyset \), and

- \( \sum_{c \in C \cup \emptyset} \gamma_{sc} = 1 \) for each student \( s \in S \)
- \( \sum_{s \in S} \gamma_{sc} = q_c \) for each school \( c \in C \)

A stochastic assignment mechanism is a function which associates to each market \((S, C, (\succeq_i)_{i \in S \cup C}, (q_e)_{e \in C})\) a stochastic assignment.

We will say that an assignment \( \rho \) weakly first-order stochastically dominates an assignment \( \omega \) if for
every student \( s \in S \), for every \( c \in C \cup \emptyset \), \( \sum_{k: k >_c c} \rho_{sk} \geq \sum_{k: k >_c c} \omega_{sk} \). An assignment \( \rho \) strongly first-order stochastically dominates an assignment \( \omega \), if \( \rho \) weakly dominates \( \omega \) and, for at least one student \( s \in S \), there exists \( c \in C \cup \emptyset \) such that \( \sum_{k: k >_c c} \rho_{sk} > \sum_{k: k >_c c} \omega_{sk} \).

A stochastic assignment mechanism \( f \) dominates a stochastic assignment mechanism \( g \) if for every market \( G \), the outcome of the first weakly dominates that of the latter, with strict dominance for at least one market.

2.2. Cycles and Interruptors

Let \( \succeq_{c, c \in C} \) be a priority structure and \( q \) a vector of school quotas. For a school \( c \in C \) and a student \( s \in S \) let \( U_s(t) = \{ s' \in S | s' \succ_s s \} \). A cycle is a six-tuple \( \alpha = (a, b, i, j, k, N) \) with distinct \( a, b \in C \) and \( i, j, k \in S \) such that the following are true:

- \( i \succ_a j \succ_a k \succ_b i \) and
- There exist (possibly empty) disjoint sets of agents \( N_a, N_b \subseteq S - i, j, k \) such that \( N_a \subseteq U_a(j), N_b \subseteq U_b(i), |N_a| = q_a - 1 \) and \( |N_b| = q_b - 1 \). Denote by \( N \) the set of all possible \((N_a, N_b)\) pairs.

A priority structure is acyclic if it contains no cycles.

If a cycle \( \alpha \) is not in a priority structure \( \succeq_{c, c \in C} \) initially, but is in the priority structure \( \succeq_{c, c \in C} \) obtained from \( \succeq_{c, c \in C} \) via a tie-breaking rule \( \tau \), we say that \( \alpha \) is an artificial cycle.

Given a problem to which the DA algorithm is applied, let \( i \) be a student who is tentatively placed at a school \( c \) at some step \( t \) and rejected from it at some later step \( t' \). If there is at least one other student who is rejected from school \( c \) after step \( t - 1 \) and before step \( t' \), that is, rejected at a step \( l \in \{ t, t + 1, \ldots, t' - 1 \} \), then we call student \( i \) an interrupter for school \( c \) and the pair \((i, c)\) an interrupting pair of step \( t' \).

3. Overview of literature on improving efficiency in school choice

3.1. Is DA ever efficient; in particular, is it efficient in large markets?

Before thinking about how to improve efficiency of DA, it is worth asking whether examining such a problem will have any impact. It is possible that as markets become large DA becomes efficient. If this is the case, because DA is typically implemented in large markets (for example, matching doctors to appointments across the country; or matching students to schools), mechanisms to fix DA's efficiency loss may have only marginal impact.

Che and Tercieux (2015) show that when student preferences over schools are uncorrelated, in the sense that students do not at all consider the same schools to be desirable, then DA is asymptotically efficient. In the school choice context, because some schools are considered 'better' and some 'worse,' this assumption seems unrealistic. Lee and Yariv (2014) consider utility as a means of measuring efficiency show that when preferences are fully aligned (that is, when students have the same preferences over schools and schools all have the same preferences over students), then DA will asymptotically produce a matching that maximizes expected average utility. Intuitively, this makes sense because if schools have the same preferences over students, no cycles can occur; the problem then becomes one of showing when stability corresponds to maximizing average expected utility when considering cardinal preferences. Again, however, in the context of school choice, because schools tend to give priority to those in their neighborhood, the assumption of fully aligned preferences is not quite realistic.

The empirical evidence on whether DA is efficient in large markets is mixed. Abdulkadiroglu, Pathak, and Roth (2008), who examine data from the New York City High School Match, which uses the deferred acceptance algorithm to match students to schools, find that 1500 students could be matched to schools they prefer over their assignment without causing priority violations, and that a further 4300 students could improve their assignment if the stability constraint were relaxed. On the one hand, this is a large number of students. On the other, these students constitute a small percentage of total students; only 1.9 and 5.5, respectively.

Hence while we are justified in looking for ways to improve DA's efficiency because lack of efficiency does affect several thousand students, it is also possible that DA is asymptotically efficient even in the context of school choice, though at the moment there is little theoretical justification for this idea.
3.2. What causes DA to incur efficiency loss?

From Example 1, we know that DA incurs efficiency loss when it runs through rejection cycles. DA incurs this efficiency loss because each student in the rejection cycle is worse off due to the presence of that cycle. In Example 1, had student \( s_1 \) not applied to \( c_1 \), \( s_1 \) would have been equally well off, and \( s_2 \) and \( s_3 \) would have been better off because the rejection cycle \( s_1 \) created would have disappeared.

It turns out that DA suffers efficiency loss only because it goes through cycles. In particular, Ergin (2002) shows that DA is Pareto efficient for all possible student preferences if and only if the priority structure of the schools is acyclic, because this is the only way to ensure that DA does not go through any rejection cycles.

Hence any attempt to remedy the efficiency loss of DA must focus on removal of cycles.

3.3. Can we improve the efficiency of DA ex-post?

Since there are a number of real students adversely affected by DA’s efficiency loss, let us say for now that improving efficiency is a goal worth working towards. Insofar as that is the case, when examining how to improve the efficiency of DA in school choice, we have two options: either we can look at how to improve efficiency before the DA algorithm is run (ex-ante), or we can look at how to improve efficiency after the DA algorithm is run (ex-post).

The earliest literature on improving efficiency in school choice looks at how to improve efficiency of DA ex-post. Erdil and Ergin (2008) seek to maintain stability, but to improve efficiency of DA by removing artificial cycles (for an example of an artificial cycle, see Example 2; there, \( \{c_1, c_2, s_3, s_1, s_2, \emptyset \} \) is an artificial cycle because it is in the priority structure created after tie-breaking, but not in the initial priority structure). In particular, Erdil and Ergin (2008) propose the use of stable improvement cycles (SIC), in which, roughly speaking, students can trade schools so long as the trade does not produce a priority violation, and thus be better off while maintaining stability. Formally,

Definition 1. Given a market and a stable matching \( \mu \), say student \( s \) desires \( c \) if \( c \succ_s \mu(s) \). Let \( B_c \) be the set of highest \( c \)-priority students among those who desire \( c \). A stable improvement cycle consists of distinct students \( s_1, ..., s_{n-1}, s_n = s_0 \) such that, for any \( k = 1, ..., n \),

- \( \mu(s_k) \in C \), (every student in the cycle is matched to a school)
- \( s_k \) desires \( \mu(s_{k+1}) \), and
- \( s_k \in B_{\mu(s_{k+1})} \)

Given a stable improvement cycle, define a new matching \( \mu' \) by

- \( \mu'(s_k) = \mu(s_k + 1) \) for all \( k \), and
- \( \mu'(s) = \mu(s) \) for all \( s \in S \) and not in \( \{s_1, ..., s_n\} \)

In Example 2, implementing a stable improvement cycle after DA produces \( \mu \) would look as follows: \( s_3 \) would desire \( c_3 \), \( s_2 \) would desire \( c_1 \), and \( s_1 \) would desire \( c_1 \). The set of highest priority students of \( c_1 \) who desire \( c_1 \) would be \( \{s_1, s_2\} \), since they have equal priority in the initial priority structure; the set of highest priority students of \( c_3 \) who desire \( c_3 \) would be \( s_3 \). Hence \( \{s_2, s_3\} \) would form a stable improvement cycle because \( s_2 \) desires \( s_3 \)'s school \( c_1 \), and \( s_2 \) is in the set of highest \( c_1 \)-priority students among those who desire \( c_1 \); and \( s_3 \) desires \( s_2 \)'s school \( c_3 \), and of those students who desire \( c_3 \), \( s_3 \) has the highest priority. So the stable improvement cycle algorithm would switch the allocated schools of \( s_2 \) and \( s_3 \), producing the student optimal stable match \( \mu' \) shown in Example 2.

Erdil and Ergin (2008) show that by repeatedly implementing stable improvement cycles, one can produce a student-optimal stable match.

The main drawback of using stable improvement cycles is that its use is not strategy proof; intuitively, a student can profitably manipulate the system by ranking a school she has high priority at high in her choices, so that she has more leverage when the stable improvement cycle mechanism is implemented.

The second post-DA efficiency-improving mechanism that has garnered much attention is the efficiency-adjusted deferred acceptance mechanism, or EADAM, introduced by Kesten (2010). Roughly speaking, this mechanism works as follows: after DA is run, identify those students who instigated a rejection cycle. Remove the school at which they started the cycle from their preference list, and re-run DA. Intuitively, because EADAM prevents the instigator of a rejection cycle from instigating that rejection cycle, it reduces the number of cycles DA runs through and hence improves the efficiency of the final match. Formally,
Definition 2. Call a student a consenting student if they agree to have a school removed from their preference list should they be an interupter for that school.

The efficiency-adjusted deferred acceptance mechanism (EADAM) works as follows:

- **Round 0:** Run the DA algorithm

- **Round $k$, $k \geq 1$:** Find the last step of the DA run in Round $k - 1$ in which a consenting interupter is rejected from the school for which he or she is an interupter. Identify all interrupting pairs of that step each of which contains a consenting interupter. If there are no such interrupting pairs, stop. For each identified interrupting pair $(i, s)$, remove $s$ from the preferences of student $i$ without changing the relative order of the remaining schools. Rerun the DA algorithm with the new preference profile.

In Example 1, $s_1$ would be an interupter for $c_1$ at step 3; since $s_1$ is the only interupter, she is certainly the last interupter, hence EADAM would (assuming $s_1$ is a consenting student) remove $c_1$ from $s_1$'s list and re-run DA, at which point there are no more interupters; hence EADAM terminates, producing the Pareto-efficient matching $\mu'$ described in Example 1.

Kesten (2010) shows that if all students consent, then EADAM produces a student-optimal matching. EADAM suffers from three drawbacks. First, it relies on the consent of students to improve efficiency. If no students consent, no efficiency gains occur. Second, EADAM can introduce priority violations (though this can be easily solved by requiring that the student an interupter $i$ kicks out of the school she interrupts at be in the same priority order for that school as $i$). Third, and perhaps most importantly, EADAM is not strategy proof—a student can, if she is sufficiently knowledgeable about preferences and priority orders, simply create a cycle at a school she desires and thereby benefit when the student who would have kicked her out no longer applies to the school, as happens under EADAM.

Given that the two most popular ex-post DA efficiency-improving mechanisms are not strategy proof, two natural question to ask are, first, is there any way to improve the efficiency of DA ex-post in a strategy proof way? And second, why even care about strategy-proofness? The answer to the first question is no; there is no way to always improve the efficiency of DA after DA has been run. In particular, Abdulkadiroglu, Pathak, and Roth (2009) show that for any tie-breaking rule, there is no strategy-proof mechanism that always results in a better matching than DA with the tie-breaking. The answer to the second question is as follows: strategy-proofness can be thought of as a form of fairness. For example, students who have parents with the discretionary time and education necessary to learn how to manipulate a mechanism may be at an advantage relative to students whose parents lack the that time or education. Hence, a mechanism that is not strategy proof does not treat all students equally, but rather may advantage those from higher socio-economic backgrounds.

Thus, it is generally considered important that a school choice mechanism be strategy proof.

3.4. Can we improve the efficiency of DA ex-ante?

We have seen that though it is possible to improve the efficiency of DA ex-post, it is not possible to dominate DA ex-post in a strategy-proof way. Is there a way to improve the efficiency of DA before the mechanism is run? This avenue of exploration seems promising because it is not constrained by the ex-post impossibility result described in the previous subsection.

The research on ex-ante improving the efficiency of DA focuses on tie-breaking. In particular, when priorities are weak, we must implement some tie-breaking mechanism before running DA. Based on the tie-breaking mechanism, each student has some probability of receiving assignment to a particular school (depending on how the tie-breaking mechanism works). For example, in Example 2 even before DA is run, we note that the tie between $s_1$ and $s_2$ at $c_1$ must be broken in some way; if we break the tie randomly, then $s_2$ would would ex-ante have a 0.5 probability of receiving assignment to $c_1$ (if the tie breaks as $s_2, s_1$ then DA would assign $s_2$ to $c_1$; on the other hand if the tie breaks as $s_1, s_2$ DA would not assign $s_2$ to $c_1$, hence overall $s_2$ has a 0.5 chance of receiving $c_1$). Hence randomized deferred acceptance can be thought of as assigning to each student a lottery over schools, and in that sense randomized deferred acceptance is a stochastic assignment mechanism.

Erdil (2014) shows that when schools have no inherent priority over students—that is, when each school is indifferent between all students—the randomized deferred acceptance mechanism can be stochastically dominated in a strategy-proof way. He does this by explicitly constructing a strategy-proof mechanism.
that improves on RDA when schools have no inherent priority order. In particular, because Erdil defines 'dominates' as an improvement for at least one market (we have used his definition in Section 2.1), he shows that he can construct a strategy-proof mechanism \( \phi \) that dominates RDA as follows: let \( \phi \) be the same as RDA in all markets except one. Now for this one selected market (which Erdil constructs), we can explicitly improve on RDA by having transferring part of one student \( s_1 \)'s probability share in a school \( c_1 \) to another student \( s_2 \), in exchange for part of that \( s_2 \)'s probability share at a school \( c_2 \) that \( s_1 \) prefers to \( c_1 \). We can construct the market and pick the transferred probability shares, in such a way that strategy-proofness is preserved for this market. Then, in the constructed market, the assignment of \( \phi \) strongly first-order stochastically dominates the assignment of RDA, hence \( \phi \) dominates RDA.

This result of Erdil (2014), that RDA can be stochastically dominated by a strategy-proof mechanism when schools have no inherent priority over students, is interesting because for the first time we have seen a way to improve on DA in a strategy proof manner; but it is constrained because it assumes schools have no priority order over students, which in reality is not generally true. For example, schools in Boston give priority to students who live near the school (Abdulkadiroglu et al., 2006). Is there a way to generalize Erdil's result?

In a sense, yes. Ehlers and Westkamp (2011) research for which priority structures it is possible to obtain a constrained efficient matching in a strategy-proof way. Ehlers and Westkamp (2011) call a priority structure solvable if it admits a constrained efficient and strategy-proof mechanism. In the context of school choice and DA, we can think of "solving" a priority structure as follows: is there a way to tie-break, even before student preferences are submitted, such that regardless of what the student preferences are, DA will produce the constrained efficient matching? The answer is: not always. Consider the following example:

**Example 3.** Let \( S = \{s_1, s_2, s_3\} \), \( C = \{c_1, c_2, c_3\} \), and priority structure given as follows:

\[
\succeq_{c_1} : s_3, \{s_1, s_2\} \\
\succeq_{c_2} : s_1, s_2, s_3 \\
\succeq_{c_3} : s_3, s_2, s_1
\]

Then this priority structure is unsolvable with respect to DA, because if we break \( c_1 \)'s indifference between \( s_1 \) and \( s_2 \) by having \( c_1 \) prioritize \( s_1 \) relative to \( s_2 \). Then if students submit the following preferences:

\[
\succ_{s_1} : c_1, c_3 \\
\succ_{s_2} : c_1, c_2 \\
\succ_{s_3} : c_2, c_1
\]

DA produces the matching \( \{(s_1, c_3), (s_2, c_2), (s_3, c_1)\} \). But all students would weakly prefer the stable matching \( \{(s_1, c_3), (s_2, c_1), (s_3, c_2)\} \), with \( s_2 \) and \( s_3 \) strictly preferring this matching. So, DA has not produced a student-optimal stable match.

On the other hand, if we break \( c_1 \)'s indifference between \( s_1 \) and \( s_2 \) by having \( c_1 \) prioritize \( s_2 \) relative to \( s_1 \). Then if students submit the following preferences:

\[
\succ_{s_1} : c_1, c_2 \\
\succ_{s_2} : c_1, c_3 \\
\succ_{s_3} : c_2, c_1
\]

DA produces the matching \( \{(s_1, c_2), (s_2, c_3), (s_3, c_1)\} \). But all students would weakly prefer the stable matching \( \{(s_1, c_1), (s_2, c_3), (s_3, c_2)\} \), with \( s_1 \) and \( s_3 \) strictly preferring this matching. So, DA has not produced a student-optimal stable match.

Since there is no way to tie-break so that DA always produces a student-optimal stable match, regardless of what preferences students submit, the above priority structure would be called unsolvable with respect to DA.

Although the priority structure in Example 3 is not solvable with respect to DA, perhaps it is nevertheless possible, given that priority structure, to dominate DA in a strategy-proof way? After all, Erdil (2014) constructed a new strategy-proof mechanism that was not DA that dominated DA. It turns out that the answer is no: there is no strategy-proof mechanism that would always produce the constrained efficient matching in Example 3 regardless of what preferences students submit. In particular, Ehlers and Westkamp
(2011) show that in order for a priority structure to be solvable, it must be acyclic, in the sense that there is some way to break ties without introducing a cycle into the induced priority structure. Example 3 is therefore unsolvable because regardless of how we break the tie between $s_1$ and $s_2$ at $c_1$, we introduce a cycle into the strict priority structure we obtain.

Erdil (2014) and Ehlers and Westkamp (2011) differ because while Ehlers and Westkamp (2011) are concerned with finding a constrained efficient and strategy-proof mechanism given a priority structure, Erdil (2014) is concerned with improving the efficiency of RDA. For example, when schools have no inherent priority order over students, Ehlers and Westkamp (2011) would say that such a priority structure is solvable—we can just tie-break by using single tie-breaking—and be done. On the other hand, Erdil (2014) would show a way to improve upon the constrained efficient matching in a strategy-proof manner. Hence while Ehlers and Westkamp (2011) is a generalization of Erdil (2014) in the sense that the former determines in which environments strategy-proof improvements upon DA can be obtained, Erdil (2014) is still distinct because it does not limit itself to constrained efficient matchings.

The result of Ehlers and Westkamp (2011) is unfortunate for improving efficiency in school choice, because most priority structures tend to be cyclic, in the sense that regardless of how one breaks ties a cycle would be introduced into the priority structure. Ehlers and Westkamp (2011) themselves point out that priority structures in which schools give priority to students who live near the school are generally unsolvable.

So, it is theoretically possible to improve upon RDA in a strategy-proof way, via the above two papers. But, how to realistically improve on RDA is still an open question, because the underlying assumptions used to show strategy-proof improvement (that school priority orders be essentially acyclic) do not hold in the real world.

3.5. Where does current research stand on the efficiency of DA, and is it desirable to extend such research?

We have seen in the previous subsections that (1) more work can be done on conditions under which DA will be asymptotically efficient, and (2) as yet, no one has found a way to realistically improve the efficiency of DA in a strategy-proof manner. What has been shown is how to improve the efficiency of DA in real-world environments, though in a not strategy-proof way, and how to improve the efficiency of DA in theoretical environments in a strategy-proof way. Given the wide-scale use of DA in school choice, it would produce tangible benefit to a large number of students if new research could find a way to improve efficiency in a strategy-proof way in realistic environments.

4. Original Results

4.1. Efficiency of DA in large markets

We try to expand the conditions under which DA will be asymptotically efficient. In particular, we improve on Che and Terceix (2014) by showing that, even when allowing students to have identical preferences over colleges, so long as these preferences are sufficiently random, DA will be asymptotically efficient. We will require the following definitions.

A random market is a tuple $\Gamma = (C, S, \succ_S, (D_s), l)$, where $l$ is the length of students' preference lists and $D_c = (p_{sc})_{j \in S}$ is a probability distribution for student $s$ over the sets of students $C$. The number of seats at any given school $S_c$ is allowed to be arbitrary. We will assume $p_{se} > 0$ for each $c \in C, s \in S$. Each random market induces a market by randomly generating preferences of each student $s$ as follows:

- **Step 1**: Select a college independently from distribution $D_s$. List this student as the top ranked college of student $s$.

In general,

- **Step $t \leq k$**: Select a college independently from distribution $D$ until a college is drawn that has not been previously drawn in steps 1 through $t - 1$. List this college as the $t^{th}$ most preferred college of student $s$.

A sequence of random markets is denoted by $(\Gamma^1, \Gamma^2, ...,)$, where $\Gamma^n = (C^n, S^n, \succ_{S^n}, (D^n_s), l^n)$ is a random market in which $|C^n| = n$ is the number of colleges. Consider the following regularity conditions.
Definition 3. A sequence of random markets \((\Gamma^1, \Gamma^2, \ldots)\) is regular if there exist positive integers \(l, \alpha, \beta,\) and \(\nu \in [0, 1]\) such that

1. \(|\Gamma^n| = l\)
2. \(|S^n| \leq \alpha n^{2\beta}\)
3. For all students \(j \in S^n\) and for all colleges \(c \in C^n, p_{jc}^{(n)} \leq \frac{\beta}{n}\)

Condition (1) says that the length of students’ preference lists is unchanged as the market grows. Condition (2) says that the number of students grows more slowly than the number of colleges. Condition (3) limits the variance in preference for colleges for a given student.

Lemma 1. Given a market \(\Gamma = (C, S, >_{SC}, (g_c)_{c \in C})\), the matching \(\mu\) produced by DA is efficient if and only if \(\Gamma\) does not contain any interrupters.

Proof. First suppose that \(\mu\) is efficient. Now suppose for contradiction that DA traversed a rejection chain. Then by the proof of Ergin, 2002 (Theorem 1), the priority structure of the reduced market is not acyclic (in the terminology of that proof, DA has gone through a ‘generalized cycle,’ and therefore the priority structure must contain a cycle), contradicting Lemma 1.

Now suppose that \(\Gamma\) contains no interrupters. Suppose for contradiction that there exists a matching \(\mu’\) such that \(\mu’(i) \geq i, \mu(i)\) for all \(i \in S\) and \(\mu’(i) > i, \mu(i)\) for some sets of students \(i \in I^* \subseteq S\). This means that each student \(i \in I^*\) was rejected from \(\mu’(i)\) during at some time during DA. Let \(t\) be the first time at which any student is rejected from their match under \(\mu’\). Now, we define student \(k\) as follows: if there is only one student rejected from their match under \(\mu’\) at time \(t\), then let \(k = t\) be that student. If there are multiple students rejected at time \(t\) from their match under \(\mu’\), pick an arbitrary college, call it \(a\), from which at least one of these students was rejected, and let \(k\) denote the student with highest priority order such that \(\mu’(k) = a\) and \(k\) was rejected from \(a\) at step \(t\). Since \(\mu’\) is a matching, we know that \(k\) is acceptable to \(a\), and therefore we must have that at step \(t\) in DA, \(k\) was rejected from \(a\), in favor of some other student, call them \(j\). Because \(\mu’\) is a matching, for \(\mu’(k) = a\), it must be the case that \(\mu’(j) \neq a\) (otherwise, the number of students matched to \(a\) would exceed \(a’s\) capacity). But since \(\mu’(j) \geq j, \mu(j)\), this means that \(\mu(j) \neq a\), so \(j\) must be rejected from \(a\) at some step \(t’ > t\), in favor of some student, call them student \(i\). Thus, \(j\) is an interrupter for school \(a\), contradicting our assumption that \(\Gamma\) contains no interrupters.

This completes the proof.

Theorem 1. Suppose that a sequence of random markets is regular. Then the probability that there are any interrupters goes to 0 as the number of colleges goes to infinity.

Proof. Recall that student \(j\) is an interrupter for some school \(a\) only if, by definition, there is some student \(k\) who \(j\) displaces at \(a\), and there is some student \(i\) who displaces \(j\) at \(a\). Therefore a necessary condition for interruption at a school \(a\) is that there are students \(i, j, k\) who all have \(a\) on their preference list. Hence, for fixed students \(i, j, k\) and school \(a\), the probability that \(j\) is an interrupter at school \(a\) is less than or equal to:

\[P_{\{i,j,k,a\}} := P(\text{Student } i, j, \text{ and } k \text{ have } a \text{ on their preference lists})\]

Now, for fixed \(a, i\), because the length of \(i\)'s preference list is \(l\), the probability that \(a\) is on \(i\)'s preference list is \(P(\text{get chosen 1st})\) or \(P(\text{get chosen 2nd})\) or...or \(P(\text{get chosen } l^\text{th})\), which is a disjoint union. Given that colleges \(c_1, \ldots, c_l\) have been selected in the first \(l\) places on a given student's preference list, the probability that college \(a\) will be selected in the \((i+1)\)th place is

\[\frac{P_a}{1 - \sum_{k=0}^l P_{c_k}} < \frac{P_a}{1 - l\frac{\beta}{n}}\]

Thus
\[ P(a \text{ is on } i's \text{ preference list}) = \sum_{i=1}^{l} P(a \text{ is selected in the } i\text{th place}) \]
\[ \leq \sum_{i=1}^{l} \frac{p_a}{1 - l\beta_n} \]
\[ = p_a \frac{l}{1 - l\beta_n} \]
\[ \leq p_a \frac{l}{1 - l\beta} \]

Because \(i, j, k\) choose their lists independently, we therefore have, for fixed students \(i, j, k\) and college \(a\) that

\[ P_{(i,j,k,a)} \leq (p_a \frac{l}{1 - l\beta})^3 \]

Thus, since \(a\) can be any college in \(C^n\), and \(i, j, k\) can be any three students in \(S^n\), the probability that any interrupters exist is:

\[ P(\text{an interrupter exists}) \leq \sum_{i,j,k \in S^n} \sum_{a \in C^n} P_{(i,j,k,a)} \]
\[ \leq \sum_{i,j,k \in S^n} \sum_{a \in C^n} (p_a \frac{l}{1 - l\beta})^3 \]
\[ \leq \left(\alpha n^\nu\right) \binom{n}{3} \left(\frac{\beta}{n} \frac{l}{1 - l\beta}\right)^3 \]
\[ \leq n^{1+3\nu} \alpha^3 \left(\frac{\beta^3 l^3}{n^3 (1 - l\beta)^3}\right) \]
\[ = 1 \]
\[ n^{2(1-\nu)} \alpha^3 \frac{\beta^3 l^3}{(1 - l\beta)^3} \]

And now, since \(\lim_{n \to \infty} \frac{1}{n^{2(1-\nu)}} = 0\), we must have \(\lim_{n \to \infty} P(\text{an interrupter exists in } \Gamma^n) = 0\), as desired. This completes the proof.

**Theorem 2.** Suppose that a sequence of random markets is regular. Then the probability that DA produces an efficient matching goes to 1 as the number of colleges goes to infinity.

**Proof.** By Lemma 1, the probability that DA produces an efficient matching is precisely the probability that no interrupters exist. By Theorem 1, this probability (that no interrupters exist) goes to 1 as the number of colleges goes to infinity. This completes the proof.

4.2. Mechanism for improving efficiency of DA in finite markets

The 'regular markets' result above can be re-formulated with more flexible conditions to show strategy proofness of a mechanism that improves the efficiency of DA. Because we have seen in our literature review that the major impediment to implementing efficiency-improving mechanisms is strategy proofness, it will be meaningful if we can show that there is a mechanism that improves efficiency of DA that is also realistically strategy proof in large markets.

We therefore present the "college efficiency-adjusted deferred acceptance mechanism", or CEADA, which may be thought of as the college-side implementation of Kesten (2010)'s EADAM mechanism. In particular, in Example 1, Kesten would have student \(s_1\) remove \(c_1\) from her preference list and re-run DA. CEADA would have \(c_1\) change its priority order to preference \(s_2\) relative to \(s_1\), and then re-run DA.
The benefit of defining the mechanism on the college side is that it will be easy to prove strategy proofness in large markets, as seen below. We will need the following definitions.

Given a market $\Gamma = (C, S, SUC, (q_c)_{c \in C})$, we say that a cycle $\alpha = (i, j, k, a, b)$, $i, j, k \in S$, $a, b \in C$, is used if at some time $t$ during the deferred acceptance algorithm, student $k$ is rejected from school $a$ in favor of student $j$; and some time $t' > t$ during DA, student $i$ is rejected from school $b$ in favor of student $k$; and at some time $t'' > t'$, student $j$ is rejected from school $a$ in favor of student $k$. We will denote the set of cycles used by DA in market $\Gamma$ by $\Gamma_\alpha$.

Assume priority orders are strict and define the college efficiency-adjusted deferred acceptance (CEADA) mechanism, as follows.

**Definition 4.** Given a market $\Gamma = (S, C, (\succ_i)_{i \in SUC}, (q_c)_{c \in C})$. Run the deferred acceptance algorithm in $\Gamma$, and construct a market $\tilde{\Gamma} = (S, C, (\succ_i)_{i \in SUC}, (q_c)_{c \in C})$ as follows:

1. Select $\alpha = (a, b, i, j, k, N) \in \Gamma_\alpha$.
2. If there exists no student $s \in S$ such that $j \succ_s a \succ_s k$, then let $k \succ_a j$.
3. Let $\preceq \equiv \tilde{\succ}$ otherwise.

Now run the deferred acceptance algorithm in $\tilde{\Gamma}$.

**Definition 5.** A sequence of random markets $(\tilde{\Gamma}, \tilde{\Gamma}^2, ...)$ is balanced if there exist positive integers $l, \alpha, \beta$, and $\nu \in [0, 1)$ such that

1. $|\tilde{\Gamma}^l| = l$
2. $|\tilde{\Gamma}^\nu| \leq an^{1/2}\nu$
3. For all students $j \in S^\nu$ and for all colleges $c \in C^\nu$, $\tilde{\Gamma}^\nu_{ji} \leq \frac{\rho}{n}$

Condition (1) says that the length of students’ preference lists is unchanged as the market grows. Condition (2) says that the number of students does not grow too much more quickly than the number of colleges. Condition (3) limits the variance in preference for colleges for a given student. Notice that these are identical to the conditions for a regular market with the exception that the growth of the number of students is more flexible; in particular, we allow there to be (many) more students than colleges, or many more colleges than students.

First, we will show that CEADA improves efficiency:

**Theorem 3.** (With notation as given in Definition 4) Using CEADA, $\mu'(s) \succeq_s \mu(s)$ for all $s \in S$, where $\mu'$ and $\mu$ are the matchings resulting from the deferred acceptance algorithm under $\tilde{\Gamma}$ and $\Gamma$, respectively.

**Proof.** If $\Gamma_\alpha = \emptyset$ or the antecedent in step (2) is not true, then $\tilde{\Gamma} = \Gamma$, so $\mu'(s) = \mu(s)$ for all $s \in S$, so $\mu'(s) \succeq_s \mu(s)$ and we are done.

If $\Gamma_\alpha \neq \emptyset$ and the antecedent in step (2) is true. To show that $\mu'(s) \succeq_c \mu(s)$ for all $s \in S$, suppose for contradiction that there exists $s \in S$ such that $\mu(s) \succ_s \mu'(s)$. Since the deferred acceptance algorithm produces a stable, hence individually rational pairing, it must be that $\mu(s) \in C$. By construction of the deferred acceptance algorithm and construction step (3) of $\tilde{\Gamma}$ (which implies that $\succ_c \equiv \tilde{\succ}$ for all $s \in S$), $s$ must have been rejected from $\mu(s)$ at some step during the deferred acceptance algorithm in $\tilde{\Gamma}$.

Let $t$ be the first step during the deferred acceptance algorithm in $\tilde{\Gamma}$ in which any student is rejected from their match under $\mu$.

Notice that by the definition of $\alpha$ being used by $\Gamma$ and the fact that $\preceq \equiv \tilde{\succ}$ except that $k \succ_a j$ whereas $j \succ_a k$, the deferred acceptance algorithm in $\Gamma$ is identical to the deferred acceptance algorithm in $\tilde{\Gamma}$ up until the step in which $j$ applies to $a$, call this step $s_1$. Hence $t \geq s_1$. We consider two cases:
• Case 1: \( t = 1 \). Since \( t \geq s_1 \), this means \( s_1 = 1 \), and hence \( j \) applies to \( a \) in step 1. Since the cycle \( \alpha = (a, b, i, j, k) \) is used by the deferred acceptance algorithm under \( \Gamma \) and by step (3) in the construction of \( \Gamma' \), college \( a \) must reject \( j \). But by definition of the deferred acceptance algorithm in \( \Gamma \) using \( \alpha \), \( j \) is also rejected from \( a \) in the deferred acceptance algorithm under \( \Gamma \), and hence \( \mu(j) \neq a \). By step (3) in the construction of \( \Gamma' \), any student \( s \neq j \) rejected in step 1 during the deferred acceptance algorithm in \( \Gamma' \) must also be rejected in step 1 of the deferred acceptance algorithm in \( \Gamma \), and hence no students are rejected in step 1 from their match under \( \mu \), contradicting the definition of \( t \).

• Case 2: \( t > 1 \). By definition of \( t \), there exists a student \( s \) who was rejected from \( c = \mu(s) \) at step \( t \) in the deferred acceptance algorithm in market \( \Gamma' \). Notice that we cannot have \( s = j \), \( c = a \) because \( \mu(j) \neq a \) as \( j \) is rejected from \( a \) in the deferred acceptance algorithm under \( \Gamma \) by definition of the algorithm using the cycle \( \alpha \). Likewise we cannot have \( s = k \), \( c = a \) because \( \mu(k) \neq a \) as \( k \) is rejected from \( a \) in the deferred acceptance algorithm under \( \Gamma \) by definition of the algorithm using the cycle \( \alpha \). Since by step (3) in the construction of \( \Gamma' \), for all except that \( k \sim_a j \) while \( j \sim_a k \), and we have shown that \( s = j, c = a \) or \( s = k, c = a \) are not possible, it must be that \( s' \geq_c s \) if and only if \( s' \geq_c s \). Because \( s \) was rejected from \( c \) at step \( t \), it must be the case that \( |\mu_t(c)| = q_0 \), and \( s' \geq_c s \) for all \( s' \in \mu_t(c) \), hence \( s' \geq_c s \) for all \( s \in \mu_t(c) \). On the other hand, because \( c = \mu(s) \), there is no step \( t' \) in the deferred acceptance algorithm under \( \Gamma \) such that \( |\mu_t(c)| = q_0 \) and \( s' \geq_c s \) for all \( s' \in \mu_t(c) \). Hence, it must be the case that there exists a student \( s' \geq_c s \) who applies to \( c \) at some step \( q \leq t \) of the deferred acceptance algorithm in \( \Gamma' \) who never applies to \( c \) during the course of the deferred acceptance algorithm in \( \Gamma' \). But the fact that \( s' \) never applies to \( c \) during the course of the deferred acceptance algorithm in \( \Gamma' \) means that \( \mu(s') \geq_c c \). Hence because \( s' \) is applying to \( c \) at step \( q \) of the deferred acceptance algorithm in \( \Gamma' \), \( s' \) must have been rejected from \( \mu(s') \) at some step \( p < q \leq t \) of the deferred acceptance algorithm in \( \Gamma' \). But this contradicts \( t \) being the first step in the deferred acceptance algorithm in \( \Gamma' \) in which any student is rejected from her match under \( \mu \).

This completes the proof.

Example 4. CEADA is not strategy-proof
Let student and school preferences be given by:

\[ \succ_{a_1}: c_1, c_2, c_3 \]
\[ \succ_{a_2}: c_1, c_2, c_3 \]
\[ \succ_{a_3}: c_3, c_1, c_2 \]
\[ \succeq_{c_1}: s_3, s_1, s_2 \]
\[ \succeq_{c_2}: s_1, s_2, s_3 \]
\[ \succeq_{c_3}: s_1, s_2, s_3 \]

Then DA does not use any cycles, and CEADA produces the same matching as DA, \( \mu = \{(s_1, c_2), (s_2, c_2), (s_3, c_3)\} \). If \( s_2 \) instead misreports her preferences as \( \succ_{a_1}: c_1, c_3, c_2 \), we are in the situation of Example 1, wherein \( (s_3, s_1, s_2, c_1, c_2) \) forms a cycle. CEADA would therefore produce matching \( \{(s_1, c_2), (s_2, c_2), (s_3, c_3)\} \). Student \( s_2 \) has therefore gained by misreporting preferences.

However, we can show that CEADA will be strategy proof in large markets under certain conditions. Because CEADA works by removing cycles, it is clear that a student can only benefit if by misreporting preferences, the student can insert herself into a cycle, as student \( s_2 \) did in the above example. Therefore it suffices to show that the probability any student can successfully insert herself into a cycle goes to zero as the market gets large.

Theorem 4. Suppose that a sequence of random markets is balanced. Then the probability that CEADA can be manipulated by a given student goes to zero as the number of colleges goes to infinity.

Proof. Recall that necessary conditions for students \( i, j, k \) and colleges \( a, b \) form a cycle used by DA are

- \( i \succ_a j \succ_b k \succ_b i \)
- Student \( i \) has both \( a \) and \( b \) on her preference list and prefers \( b \) to \( a \); student \( k \) has both \( a \) and \( b \) on her preference list and prefers \( a \) to \( b \); student \( j \) has \( a \) on her preference list.
Hence the probability that fixed $i, j, k \in S$, $a, b \in C$ form a cycle is less than or equal to:

$$P_{(i,k,a,b)} := P(\text{Student } i \text{ has both } a \text{ and } b \text{ on her preference list and prefers } b \text{ to } a;$$

$$\text{ student } k \text{ has both } a \text{ and } b \text{ on her preference list and prefers } a \text{ to } b;$$

$$\text{ student } j \text{ has } a \text{ on her preference list})$$

Now, for fixed $a, i$, because the length of $i$'s preference list is $l$, the probability that $a$ is on $i$'s preference list is $P(\text{get chosen 1st}) \text{ or } P(\text{get chosen 2nd}) \text{ or ... or } P(\text{get chosen } l\text{th})$, which is a disjoint union. Given that colleges $c_1, \ldots, c_l$ have been selected in the first $i$ places on a given student's preference list, the probability that college $a$ will be selected in the $(i + 1)$th place is

$$\frac{p_a}{1 - \sum_{k=0}^{i-1} p_{c_k}} \leq \frac{p_a}{1 - l \frac{\beta}{n}}.$$ 

Thus

$$P(\text{a is on i's preference list}) = \sum_{i=1}^{l} P(\text{a is selected in the } i\text{th place})$$

$$\leq \sum_{i=1}^{l} \frac{p_a}{1 - l \frac{\beta}{n}}$$

$$= p_a \frac{l}{1 - l \frac{\beta}{n}}$$

$$\leq p_a \frac{l}{1 - l \beta}$$

and

$$P(\text{b is on i's preference list given that a is on i's preference list}) \leq p_b \frac{l}{1 - l \beta}$$

by identical analysis. Because $i, j,$ and $k$ choose their lists independently, we therefore have, for fixed students $i, k$ and colleges $a, b$ that

$$P_{(i,k,a,b)} \leq (p_a \frac{l}{1 - l \beta})^2 (p_b \frac{l}{1 - l \beta})^2.$$ 

Now notice that under CEADA, a student only benefits if she is the injured party, that is, if she is in the position of student $k$, who is rejected from college $a$ in favor of $j$. Thus, for a fixed student $k$, the probability that she can successfully insert herself into a cycle is

$$P(\text{k can successfully insert self into a cycle in market } \tilde{\Gamma}^n) \leq \sum_{i,j \in S^n} \sum_{a,b \in C^n} P_{(i,k,a,b)}$$

$$\leq \sum_{i,j \in S^n} \sum_{a,b \in C^n} (p_a \frac{l}{1 - l \beta})^2 (p_b \frac{l}{1 - l \beta})^2$$

$$\leq \left( \alpha \frac{\beta}{2} \right)^n \frac{n!}{2^n} \left( \frac{\beta}{n} \frac{l}{1 - l \beta} \right)^5$$

$$< n^{2+2 \alpha \frac{\beta}{n}} \alpha^3 \left( \frac{\beta}{n} \frac{l}{1 - l \beta} \right)^5$$

$$\leq \frac{1}{n^{3(1-\nu)}} \alpha^3 \frac{\beta^5 l^5}{(1 - l \beta)^5}$$

And now, since $\lim_{n \to \infty} \frac{1}{n^{3(1-\nu)}} = 0$, we must have $\lim_{n \to \infty} P(\text{a fixed student can insert herself into a cycle in } \tilde{\Gamma}^n) = 0$, as desired. This completes the proof.
5. Conclusion

In this paper we have summarized the main research on efficiency in school choice. The research on whether efficiency is a theoretical concern in large markets is limited; so far, it has been shown that when student preferences are uncorrelated in that students do not have identical 'tiers' for different schools, DA will be asymptotically efficient. Two popular ex-post mechanisms, SIC (proposed by Erdil and Ergin, 2008) and EADAM (proposed by Kesten, 2010) improve efficiency but are not strategy-proof. Ex-ante strategy-proof improvements on DA are possible but only in limited environments, none of them realistic in the context of school choice. Recognizing that further research would be helpful, given DA’s prominence not only in school choice but in many other markets, we present original results on asymptotic efficiency and strategy-proof improvement mechanisms. In particular, we show that under sufficiently random preferences, and when the number of positions is greater than the number of students, DA will be asymptotically efficient. We further present a mechanism, CEADA, that improves on Kesten (2010), in that CEADA is non-manipulable under somewhat realistic conditions (in large markets with sufficiently random preferences).
Papers cited


