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Deterrence and the Optimal Use of Prison, Parole, and Probation

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Abstract: In this article we derive the mix of criminal sanctions—choosing among prison, parole, and probation—that achieves any target level of deterrence at least cost. We assume that prison has higher disutility and higher cost per unit time than parole and probation and that potential offenders discount the future disutility of sanctions at a higher rate than the state discounts the future costs of sanctions. Our primary insight is that there is a "front-loading advantage" of imprisonment due to these differential discount rates. This advantage implies that (a) whenever a sentence includes both a prison term and a parole term, the prison term should be imposed first; and (b) it may be optimal to employ a prison term even if prison has higher cost per unit of disutility than parole and probation and even if prison is not needed to achieve the target level of deterrence.

Key words: crime; imprisonment; parole; probation; prison costs; deterrence; sanctions

JEL codes: H23; K14; K42

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1. Introduction

Although imprisonment is widely employed in the United States, the use of parole and probation—in which offenders are supervised outside of prisons—is far greater. For example, at the end of 2016, there were 2.16 million individuals incarcerated in local, state, and federal correctional facilities, and 4.54 million individuals subject to parole or probationary supervision.[1] Parole and probation are also common sanctions in many other countries.[2] It seems surprising, therefore, that while there is a voluminous literature in which the deterrent effect of imprisonment is considered,[3] examination of the deterrent effect of parole or probation has been virtually absent.[4] The contribution of our article is to provide a theoretical analysis of the merits of parole and probation as means of achieving desirable deterrence, together with, or as an alternative to, imprisonment. We undertake this task by deriving the mix of criminal sanctions—chosen from among prison, parole, and probation—that achieves any target level of deterrence at least cost.[5]

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[1] See Kaeble and Cowhig (2018, p. 2, Table 1). The vast majority—81 percent—of those supervised outside of prison are on probation (generally without having served any time in prison).

[2] For example, France had 60,896 individuals incarcerated in 2015 and 130,163 on probation or parole at the end of 2015. The corresponding statistics for England and Wales were 85,843 individuals incarcerated and 99,940 on probation or parole. See Walmsley (2016, pp. 10-11) and Aebi and Chopin (2016, pp. 18-19, Table 1.1, columns 1.2.1-1.2.3 and 1.2.9). In the second quarter of 2018, Australia had 42,855 individuals incarcerated and 69,397 in "community-based corrections." See Australian Bureau of Statistics (2018).

[3] See, for example, the survey articles by Levitt and Miles (2007), Polinsky and Shavell (2007), and Chalfin and McCrary (2017).

[4] For instance, in the surveys of the economics of enforcement and criminal punishment mentioned in the preceding footnote, a total of 504 articles are cited, of which only three, all unpublished, concern parole or probation. There are two relevant omissions from these surveys—Miceli (1994) and Garoupa (1997)—that we discuss below.

[5] There are other functions that parole can serve that we do not consider here, notably, reducing the cost of incapacitating offenders after it has been determined that they pose a low risk of recidivism, or providing a reward to prisoners who behave well. See, for example, Bernhardt et al. (2012) and Polinsky (2015).

Parole and probation are both forms of out-of-prison supervision of offenders, where parole follows a prison term and probation is a stand-alone sanction. While the degree of state supervision need not be the same under parole and probation, nor their costs, we will treat parole and probation as equivalent sanctions per unit time.[6] Thus, their difference in our analysis will be purely semantic—when a period of out-of-prison supervision is combined with a prison term, we will refer to this period as parole, but when such a period is used by itself, we will refer to it as probation.

In our model, prison imposes higher disutility on offenders and generates higher costs for the state per unit time than do parole and probation. However, the cost of prison per unit of disutility can be lower or higher than the cost of parole and probation per unit of disutility.[7] We assume that potential offenders discount the future disutility of sanctions and that the state discounts the future costs of sanctions.

We focus on the realistic case in which offenders discount the disutility of sanctions at a higher rate than the state discounts the costs of sanctions.[8] Our primary insight is that there is a "front-loading advantage" of imprisonment over parole and probation due to these differential discount rates. Specifically, because prison imposes disutility at a higher level per unit time than do parole and probation—prison front-loads disutility—a shorter term can be used to achieve any given level of deterrence. Everything else equal, a shorter term will reduce the deterrence-diluting effect of offender discounting of disutility, which is beneficial. But a shorter term also

[6] Although in practice parole is not usually referred to as a *sanction*, as prison and probation are, we will, for economy of language, refer to it as such here.

[7] We discuss some empirical evidence regarding the disutility and cost of prison and out-of-prison supervision in Section 5 below, and conclude that out-of-prison supervision is likely to be more cost effective.

[8] See the first paragraph in Section 4 below on why we believe this to be the realistic case.

will reduce the cost-diluting effect of the state's discounting of costs, which is detrimental. When the offender's discount rate exceeds the state's discount rate, the first effect dominates the second, resulting in a net advantage from using imprisonment.

We demonstrate that the front-loading advantage of imprisonment has two significant implications. First, whenever a sentence employs both a prison term and a parole term, the prison term should be imposed first. Second, even if prison is less cost effective than probation—has a higher undiscounted cost per unit of disutility—and even if prison is not needed to achieve the deterrence target, it may be optimal to employ a prison term.

Section 2 presents the basic model and Section 3 derives the optimal sentence in the case in which the offender's and the state's discount rates are equal, providing a benchmark for assessing the effects of differential discount rates. Section 4 analyzes the main case of interest, in which offenders discount disutility at a higher rate than the state discounts cost. Section 5 concludes with several comments. Proofs of the results are contained in the Appendix.

Although this is the first article to compare the use of prison, parole, and probation to promote deterrence when offenders discount disutility and the state discounts costs, Miceli (1994) and Garoupa (1997) are of related interest. They too examine parole and probation along with imprisonment, but they do not account for discounting and therefore cannot derive the results emphasized here.[9] Additionally, Polinsky and Shavell (1999) and McCrary (2010) incorporate discounting in their analyses of imprisonment sanctions, but do not investigate the use of parole or probation as a supplement or alternative to imprisonment.

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[9] However, unlike us, they consider the use of parole to reward prisoners for good behavior and the opportunity of offenders to commit additional crimes during parole or probation.

2. The State's Problem

The offender's sentence begins at time $t = 0$. At any time $t \geq 0$, the state can impose a prison sanction or an out-of-prison supervision sanction or no sanction at all. We will refer to the last choice as imposing the *null sanction*. The prison sanction inflicts greater disutility on an offender and causes the state to incur higher cost per unit time than does the out-of-prison supervision sanction. The null sanction generates zero disutility and zero cost. The three possible sanctions will be numbered in decreasing order of severity—denoting prison with 1, out-of-prison supervision with 2, and the null sanction with 3. Thus, let

θ_i = disutility of sanction i per unit time, where $\theta_1 > \theta_2 > \theta_3 = 0$;[10] and

c_i = cost of sanction i per unit time, where $c_1 > c_2 > c_3 = 0$.

A *sentence* is a function $\sigma(t)$ mapping each non-negative time to a sanction, such that the value of $\sigma(t)$ changes only a finite number of times and sanctions are imposed for strictly positive lengths of time. Let Σ be the set of sentences. A *term* of sentence $\sigma(t)$ is an interval of time such that $\sigma(t)$ is constant throughout the interior of the interval but changes at the lower bound of the interval if the lower bound is positive and at the upper bound of the interval if the interval is finite. Terms are thus the longest periods of time over which a sentence imposes a constant punishment, and this definition corresponds with the usual meaning of, for example, "a term of prison." By definition, every sentence $\sigma(t)$ partitions the set of non-negative times into a finite number of terms.

As noted in the introduction, if an out-of-prison supervision term is employed in combination with a prison term it will be referred to as a parole term, whereas if it is used alone,

[10] We discuss in Section 5 below why we believe that our principal results would not be affected if the disutility from sanctions were not constant per unit time, as is assumed here.

it will be referred to as a probation term.

Potential offenders discount the future disutility of sanctions and the state discounts the future costs of sanctions. Let

r = rate at which potential offenders discount the disutility of sanctions; $r > 0$;[11] and

ρ = rate at which the state discounts the cost of sanctions; $\rho > 0$.

We assume for analytical convenience that individuals live forever.[12]

The state seeks to achieve a target level of deterrence, which is to say to impose some specified level of discounted disutility on offenders through its choice of sanctions. Let

k = target level of deterrence; $k > 0$.

We assume that k is strictly less than the present value of the disutility that would result from a perpetual prison term, that is, that $k < \theta_1/r$. Otherwise, if $k = \theta_1/r$, the only sentence that could achieve the target level of deterrence would be a perpetual prison term, or if $k > \theta_1/r$, no sentence could achieve the target level of deterrence.

The state's problem is to choose a sentence $\sigma(t)$ from the set of sentences Σ that achieves the target level of deterrence k at the lowest discounted cost.[13] Thus, using the notation $c(\theta_i) = c_i$, the state's problem is:

$$\min_{\sigma(\cdot) \in \Sigma} \int_0^\infty c(\sigma(t))e^{-\rho t}dt \tag{1}$$

subject to

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[11] See the third comment in Section 5 for a discussion of hyperbolic discounting.

[12] Since the optimal terms of prison and out-of-prison supervision will be seen to be finite in the case on which we focus in Section 4 (in which offenders discount disutility at a higher rate than the state discounts costs), this assumption does not affect our main results.

[13] The state's problem could be formulated more generally as maximizing social welfare, defined as the utility of offenders from committing harmful acts, less the harm done, less the cost of catching offenders, and less the social cost of sanctions. Clearly, a necessary condition for a social-welfare maximizing enforcement policy is that the disutility imposed on offenders through sanctions be achieved at the lowest possible cost to the state.

$$\int_0^\infty \sigma(t) e^{-rt} dt = k. \qquad (2)$$

The state's problem as just described does not impose any restrictions on the number of terms that each sanction can be used (aside from ruling out an infinite number of terms). In the case that we focus on, in which the offender's discount rate exceeds the state's discount rate, we prove in the Appendix that the optimal sentence employs a prison term and an out-of-prison supervision term at most once each. In the benchmark case, in which the discount rates are equal, the optimal sentence is not unique, but the set of optimal sentences includes a sentence that uses at most one term of each sanction. Accordingly, in describing our results in the following sections, we will refer to a single prison term and a single out-of-prison supervision term.[14] Let

s_1 = length of prison term; $s_1 \geq 0$; and

s_2 = length of out-of-prison supervision term (parole or probation); $s_2 \geq 0$.

3. Optimal Sentences When Offenders Discount Disutility at the Same Rate at Which the State Discounts Costs

To better understand the effect of differential discounting on the optimal choice of sanctions, we derive here the optimal sentence when the offender's disutility discount rate and the state's cost discount rate are equal, $r = \rho$. In this case there is no front-loading advantage of imprisonment since the benefit of imposing disutility early is exactly offset by the detriment of bearing costs early. Hence, as we will show, there is no reason to use imprisonment before parole if both sanctions are employed and no reason to use a prison term unless it is more cost

[14] For brevity, when we list the terms of a specific sentence, we generally omit terms of the null sanction.

effective than parole and probation or is needed to achieve the deterrence target. In sum, the most cost-effective sanction—the one with the lowest cost per unit of disutility—should be used to the greatest extent feasible.

In the present case, optimal sentences generally are not unique. For instance, a prison term beginning at $t = 0$ that achieves the target level of deterrence is equivalent to a longer prison term starting at some $t > 0$ that also achieves the target. For expositional simplicity, we will only report optimal sentences that begin with a term of imprisonment or out-of-prison supervision that starts at $t = 0$, provided that such a sentence exists.[15]

Proposition 1: If offenders discount disutility at the same positive rate that the state discounts costs, $r = \rho > 0$, and

(a) if $c_1/\theta_1 < c_2/\theta_2$, then for any target level of deterrence $k \in (0, \theta_1/r)$, a finite prison term

$$s_1 = (1/r)ln[\theta_1/(\theta_1 - rk)] > 0 \tag{3}$$

is optimal;

(b) if $c_1/\theta_1 = c_2/\theta_2$, then for any target level of deterrence $k \in (0, \theta_1/r)$, any mix of sanctions that satisfies the deterrence constraint is optimal; and

(c) if $c_1/\theta_1 > c_2/\theta_2$, then

(i) for a relatively low target level of deterrence $k \in (0, \theta_2/r)$, a finite probation term

$$s_2 = (1/r)ln[\theta_2/(\theta_2 - rk)] > 0 \tag{4}$$

is optimal;

(ii) for an intermediate target level of deterrence $k = \theta_2/r$, an infinite probation term $s_2 = \infty$ is optimal; and

[15] The Appendix provides the more general results.

(iii) for a relatively high target level of deterrence $k \in (\theta_2/r, \theta_1/r)$, both a finite prison term

$$s_1 = (1/r)ln[(\theta_1 - \theta_2)/(\theta_1 - rk)] > 0 \tag{5}$$

followed immediately by an infinite parole term $s_2 = \infty$, and a finite parole term

$$s_2 = (1/r)ln[(\theta_2 - \theta_1)/(\theta_2 - rk)] > 0 \tag{6}$$

followed immediately by an infinite prison term $s_1 = \infty$, are optimal.

Comments: (a) In essence, Proposition 1 states that the sanction with the lower *undiscounted* cost per unit of *undiscounted* disutility, c_i/θ_i, should be relied upon to the greatest extent possible. Since the offender's disutility discount rate equals the state's cost discount rate, this implies that the sanction with the lower *discounted* cost per unit of *discounted* disutility will be given primacy.

(b) If imprisonment has the lower cost per unit of disutility, a prison term can be relied upon exclusively regardless of the target level of deterrence (up to the maximum of θ_1/r), due to prison being the more potent sanction. This explains part (a) of the proposition.

(c) If out-of-prison supervision—parole or probation—has the lower cost per unit of disutility, ideally a probation term would be used. But because probation has lower potency than prison, a probation term can achieve the target level of deterrence only if the target is not too high (not exceeding θ_2/r). If the target level of deterrence exceeds the level of deterrence achievable by probation, then some prison time will be required. But when parole has a lower cost per unit of disutility than prison, it will be optimal to minimize the contribution of the prison term to the present value of disutility. This can be accomplished equally well by employing prison first and then switching to parole as soon as possible, while still satisfying the deterrence constraint, or by using parole first and switching to prison only when necessary in order to meet the deterrence constraint. This explains the two alternative, equally desirable, options in part (c) of the proposition.

(d) It is obvious from the preceding discussion that when prison and out-of-prison supervision have the same cost per unit of disutility, the choice of sanctions is immaterial provided that the deterrence target is achieved.

4. Optimal Sentences When Offenders Discount Disutility at a Higher Rate Than the State Discounts Costs

We now turn to what we believe is the most realistic case, when offenders discount disutility at a higher rate than the state discounts costs. There is empirical support for the primacy of this case. Mastrobuoni and Rivers (2016) estimated criminal discount rates at 30 percent.[16] Åkerlund et al. (2016) observed that individuals with high discount rates—58 percent or higher—were significantly more likely to participate in criminal acts; they concluded that "[o]ur findings are therefore consistent with the idea that criminals have extremely high discount rates." In the same vein, Lee and McCrary (2017) concluded that their model and data supported the view that "offenders [have] short time horizons, leading them to perceive little difference between nominally long and short incarceration periods."[17] Studies of public discount rates, in contrast, have found rates generally to be between 3.5 percent and 7 percent.[18]

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[16] In their terminology, they found an annual discount factor of 0.74, which we have converted to a continuous discount rate.

[17] Earlier discussions of the discount rates of criminals also came to the conclusion that such rates were unusually high, especially among younger offenders. See, for example, Wilson and Hernnstein (1985, pp. 204-05).

[18] See, for example, Burgess and Zerbe (2011) and Moore et al. (2013).

We begin by showing that in the present case if both prison and parole are employed, it is optimal to use prison first, and we then derive the optimal sentence given the sanctions' relative cost per unit of disutility and the target level of deterrence.

Proposition 2: If offenders discount disutility at a higher rate than the state discounts costs, $r > \rho > 0$, then a prison term should be used before a parole term if both sanctions are employed in an optimal sentence.

Comments: (a) The intuition underlying this result is easiest to see if the state's discount rate is assumed to be zero. Consider a parole term followed by a prison term that together achieve a certain level of deterrence. By reversing the order of the terms, with prison now used first, deterrence will rise because the sanction that produces the higher level of disutility per unit time now occurs earlier and is diluted less by offender discounting. Given the assumption that the state's discount rate is zero, the state's costs are unaffected. Since deterrence is higher, one or both of the terms can be shortened to restore the original level of deterrence. This will result in lower costs, implying that it is preferable to employ prison before parole. This intuition applies whenever the offender's disutility discount rate exceeds the state's cost discount rate, including when the state's discount rate is positive.[19]

(b) The present result is the first to provide an explanation grounded in deterrence theory of why a prison term should precede a parole term. As observed in the previous section, if the offender's and the state's discount rates were the same, the order would not matter (see Proposition 1(c)(iii)).

[19] It is worth noting in passing that, were the state's discount rate higher than the offender's, it would be optimal to use parole before prison.

(c) In practice, parole terms do follow prison terms when both sanctions are employed. This is true in the United States as well as in every other country's criminal justice system with which we are familiar. While there are other reasons why this should be so,[20] the result of Proposition 2 provides at least a partial basis for this practice.

Example: To illustrate Proposition 2, suppose that the offender's disutility discount rate is fifteen percent and the state's cost discount rate is two percent. Assume that prison imposes $30,000 of disutility and costs $30,000 per year, while parole imposes $2,500 of disutility and costs $2,000 per year.

Let sentence A consist of one year of parole followed by one year of prison, resulting in discounted disutility of $26,299 and discounted cost of $31,094. Now reverse the order of the terms in sentence A, so that the one-year prison term occurs before the one-year parole term. This change results in a sentence in which the present value of disutility is higher than in sentence A because the more severe sanction, prison, now occurs first. Let sentence B consist of a prison term of *less than one year* followed by a parole term of greater than one year such that the sum of the two terms remains at two years but the present value of disutility is restored to the level created by sentence A. In this example, the length of the new prison term that equates the disutility generated by sentences A and B in this way is 0.85 years.

Sentence B's partial substitution of parole for prison tends to reduce its cost relative to that of sentence A, but sentence B's use of the more costly sanction first creates a countervailing effect. When the offender's disutility discount rate is higher than the state's cost discount rate, the first effect dominates the second, resulting in a lower discounted cost. Specifically, the

[20] Both of the rationales for parole mentioned in note 5 above require that parole follow prison.

discounted cost of sentence B is \$27,558, while that of sentence A was seen to be \$31,094. Hence, sentence A, in which parole was used before prison, cannot be optimal.

In the present case the following proposition describes the optimal sentence. Unlike in the benchmark case of the previous section, optimal sentences are unique and begin with a term of imprisonment or out-of-prison supervision.

Proposition 3: If offenders discount disutility at a higher rate than the state discounts costs, $r > \rho > 0$, and

(a) if $c_1/\theta_1 \leq c_2/\theta_2$, then for any target level of deterrence $k \in (0, \theta_1/r)$, a finite prison term

$$s_1 = (1/r)ln[\theta_1/(\theta_1 - rk)] > 0 \tag{7}$$

is optimal; and

(b) if $c_1/\theta_1 > c_2/\theta_2$, then

(i) for a relatively low target level of deterrence $k \in (0, \kappa]$, where

$$\kappa = (\theta_2/r) - (\theta_2/r)\{(c_2/\theta_2)[(\theta_1 - \theta_2)/(c_1 - c_2)]\}^{r/(r-\rho)} < \theta_2/r, \tag{8}$$

a finite probation term

$$s_2 = (1/r)ln[\theta_2/(\theta_2 - rk)] > 0 \tag{9}$$

is optimal; and

(ii) for a relatively high target level of deterrence $k \in (\kappa, \theta_1/r)$, a finite prison term

$$s_1 = (1/r)ln[(\theta_1 - r\kappa)/(\theta_1 - rk)] > 0 \tag{10}$$

followed immediately by a finite parole term

$$s_2 = (1/r)ln[\theta_2/(\theta_2 - r\kappa)] > 0 \tag{11}$$

is optimal.

Comments: (a) As discussed in the introduction, when the offender's disutility discount rate exceeds the state's cost discount rate, there is a front-loading advantage of prison over parole or probation. This is because the advantage of imposing greater disutility sooner through a prison term, and thereby reducing the deterrence-diluting effect of offender discounting, more than offsets the disadvantage of bearing greater costs sooner.

(b) If prison is weakly cheaper per unit of disutility than out-of-prison supervision ($c_1/\theta_1 \leq c_2/\theta_2$), prison should be used without parole both because prison is at least as cost effective at any point in time and it has a front-loading advantage. Moreover, since prison is more potent than out-of-prison supervision, it can achieve any target level of disutility (up to θ_1/r), so there is no reason to consider out-of-prison supervision in order to satisfy the deterrence constraint. This reasoning explains part (a) of the proposition.

(c) If prison is more expensive per unit of disutility than out-of-prison supervision ($c_1/\theta_1 > c_2/\theta_2$), there is a tradeoff in the choice of the sanctions. Although out-of-prison supervision then would be more cost effective at any point in time, prison still has a front-loading advantage. This advantage increases with the target level of deterrence k, since a higher k will require longer terms of sanctions, which will augment the deterrence-diluting effect of offender discounting more than the cost-diluting effect of state discounting. Hence, when the target level of deterrence is relatively low (up to the threshold level κ), the front-loading advantage of imprisonment will be dominated by the superior cost effectiveness of out-of-prison supervision, making it optimal to rely on a probation term. But if the target level of deterrence is relatively high (exceeding κ), the front-loading advantage of imprisonment will make imprisonment worth employing despite its higher cost per unit of disutility. This reasoning explains the general thrust of part (b) of the proposition, though we have two more particular points about this part that we now turn to.

(d) It is noteworthy that the threshold level of deterrence κ, at and below which probation is optimal and above which prison followed by parole is optimal, is strictly less than θ_2/r, the maximum level of deterrence that can be achieved by probation. This implies that there is a range of the target level of deterrence over which, even though prison is more costly per unit of disutility than probation, *and even though probation is capable of achieving the target level of deterrence*, it is desirable to use a prison term (followed by a parole term). The explanation, of course, is the one provided in the previous paragraph—that imprisonment has a front-loading advantage that can offset its lower cost effectiveness.

(e) One other point of interest regarding part (b) of the proposition is that when prison and parole are used together, only the prison term is lengthened as the target level of deterrence k increases. This result is explained by the fact, observed above, that the front-loading advantage of imprisonment grows as the target level of deterrence increases. In contrast, the disadvantage of imprisonment in terms of its cost effectiveness—that it has a higher undiscounted cost per unit of disutility at each point in time (by assumption in part (b))—remains constant. Hence, there will be a critical value of the target level of deterrence, κ, at which the front-loading advantage of prison will just offset its cost-effectiveness disadvantage. For all higher levels of deterrence, it will be increasingly beneficial to use prison rather than parole.

Example: In the example employed above, the cost per unit of disutility is \$0.80 for out-of-prison supervision (= \$2,000/2,500) and \$1.00 for prison (= \$30,000/30,000). Hence, part (b) of Proposition 3 is applicable. The optimal mix of sanctions then depends on whether the target level of deterrence k is below or above the threshold level of deterrence, κ = \$4,048, which is lower than the level of deterrence achievable by a sentence of lifetime probation, θ_2/r = \$16,667.

If, for example, k = \$2,000 < κ, the optimal sentence is a probation term of 0.85 years.

Suppose instead that k = \$10,000, which is between κ and θ_2/r. Thus, probation, which has a lower cost per unit of disutility than does prison, would be capable of achieving the target level of deterrence. Specifically, a probation term of 6.11 years would accomplish this. Nonetheless, the optimal sentence is a prison term of 0.21 years followed by a parole term of 1.86 years.

If, say, k = \$100,000 > θ_2/r, some imprisonment is needed in order to satisfy the deterrence constraint. In this case, the optimal sentence is a prison term of 4.48 years followed by a parole term of 1.86 years. For the reason explained in comment (e) above, only the prison term has increased from the previous case.

5. Concluding Comments

The relative cost effectiveness of sanctions: As observed in Proposition 3, the optimal mix of sanctions depends on their cost per unit of disutility, c_i/θ_i—that is, on their cost effectiveness. If $c_1/\theta_1 \leq c_2/\theta_2$, that is, prison is at least as cost effective as out-of-prison supervision, then only prison terms should be used, whereas if $c_1/\theta_1 > c_2/\theta_2$, out-of-prison supervision should be used exclusively or in combination with prison, depending on the target level of deterrence.

The average cost of incarceration in the United States (including jails and prisons) is \$28,835 per year, as of 2009.[21] The corresponding average cost of out-of-prison supervision is

[21] See Pew Center on the States (2009, p. 12), reporting an average cost among the states surveyed of \$79 per inmate per day. The vast majority of inmates are in state jails and prisons, not Federal institutions.

$1,529 per year.[22] Since the cost of out-of-prison supervision is only 5.3 percent of that of prison, out-of-prison supervision will be more (less) cost effective than prison if the disutility of out-of-prison supervision is greater (less) than 5.3% of that of prison.

Several attempts have been made to assess the disutility of prison relative to out-of-prison supervision. For example, based on prisoners' perceptions of the severity of sanctions, Spelman (1995, p. 121) concluded that one year in jail or prison was "roughly equal" to five or six years of probation and that three months in jail or prison were approximately equivalent to one or two years of probation.[23] Assuming an offender discount rate of 15 percent and using the mid-points of the probation ranges, the first comparison implies that the disutility from probation is equal to approximately 25 percent of that of prison, and the second comparison implies an 18 percent result.[24]

Hence, in light of the condition stated two paragraphs above, out-of-prison supervision is much more cost effective than prison. Relying on different sanction cost data, Spelman (1995, p. 127) similarly concluded that "community sanctions" (including probation) were the "most efficient" punishments and that "[p]risons are least efficient of all." Thus, part (b) of Proposition 3 is likely to be the more applicable part, in which case probation terms are optimal for relatively minor crimes and increasingly severe prison terms followed by a fixed parole term are optimal for more significant crimes.

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[22] See Pew Center on the States (2009, p. 12), reporting an average cost of $3.42 per day for probationers and $7.47 per day for parolees. The number used in the text is based on a weighted average of these numbers, with 81% of those supervised out of prison being on probation (see note 1 above).

[23] He attributes the latter result to Petersilia and Deschenes (1994, p. 6).

[24] These numbers were calculated by assuming that the disutility of out-of-prison supervision is some multiple λ of the disutility of prison and finding the value of λ that equates the present value of disutility from a prison term of length s_1 and the present value of a probation term of length s_2. If the discount rate were higher than 15 percent, the numbers reported in the text would be higher.

Non-constant disutility of sanctions: We have been assuming that the disutility per unit time of a sanction, θ_i, is constant. It may well be the case, however, that this disutility varies with the time elapsed since the start of the term. For instance, the first year in prison may generate a higher level of disutility than subsequent years due, say, to adaptation to the prison environment. Or the incremental disutility might increase if a prisoner has young children at home from whom he grows increasingly distant.

We do not believe that allowing for the non-constant disutility of sanctions would fundamentally affect our results provided that, as seems reasonable, the disutility from a year in prison always exceeds the disutility from a year of out-of-prison supervision. For then, if offenders discount at a higher rate than the state, there would still be a front-loading advantage of imprisonment, providing a reason to use prison before parole and to use prison even when prison is less cost effective than probation.

The main effect of non-constant disutility of sanctions would be on the optimal lengths of sanctions. If, for instance, the marginal disutility of imprisonment declines over time, then when a prison term is combined with a parole term, the switch from imprisonment to parole would tend to occur sooner.

Hyperbolic discounting by potential offenders: Although we have not analyzed the issues addressed in this article under the assumption that potential offenders discount the disutility of sanctions hyperbolically, we do not believe that that our main results would be affected by such an assumption for the following reason. These results, developed in Section 4, stem from the assumption that an offender's discount rate exceeds that of the state. Under hyperbolic discounting, an individual's discount rate over a short time horizon exceeds his discount rate over a longer time horizon. We conjecture that as long as the latter discount rate is greater than the state's discount rate, then the qualitative results derived in Section 4 would continue to hold.[25] The empirical evidence concerning criminals' discount rates that we discussed in the previous section supports the view that even their long-term discount rates are far higher than the state's discount rate.[26]

Discount rates differ by age: It is widely believed that younger offenders, especially young males, tend to discount future utility at a rate that is significantly higher than the average for the population as a whole.[27] Suppose that there are two groups of offenders, young ones and older ones, with the former group having a higher discount rate r. We assume that both values of r exceed the state's discount rate ρ, so that the analysis in Section 4 applies.

If prison is weakly more cost effective than out-of-prison supervision, $c_1/\theta_1 \leq c_2/\theta_2$, then a prison sanction would be optimal for both groups of offenders, but the term would have to be longer for the younger offenders due to their higher discount rate.

If prison is less cost effective than out-of-prison supervision, $c_1/\theta_1 > c_2/\theta_2$, then the optimal sentence depends on whether the target level of deterrence is below or above a threshold (see Proposition 3(b)). This threshold will be lower the higher is the offender's discount rate r.[28] Hence, there are three possibilities—that the target level of deterrence is below the threshold for

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[25] If the discount rate were to decline continuously with time and approach zero, then, of course, the premise of this conjecture could not hold. But this characterization of hyperbolic discounting is inconsistent with the findings of Frederick et al. (2002, p. 361) from a review of numerous empirical studies of discounting.

[26] See the Mastrobuoni and Rivers (2016) and Åkerlund et al. (2016) articles. They estimate criminals' discount rates over an extended period of time assuming a constant discount rate.

[27] For example, Wilson and Herrnstein (1985, p. 205) argue that young people have higher criminal tendencies than older people due in part to their generally higher discount rates. Lee and McCrary (2017) find evidence consistent with high discount rates in their young and predominantly male sample of offenders.

[28] Recall that below the threshold κ, probation is used, while above κ, prison and parole are employed. As r rises, the front-loading advantage of imprisonment rises, making it more desirable, everything else equal, to impose a sentence consisting of prison and parole rather than of probation. Equivalently, κ declines.

both groups; that it is between the two thresholds; or that it exceeds both thresholds.

The results in the first case are straightforward; both groups should be subject to probation sanctions, with the younger group receiving a longer term due to its higher discount rate. In the third case, both groups should be subject to a prison sanction followed by a parole sanction, with the younger group receiving a longer prison term and a shorter parole term because its higher discount rate strengthens the front-loading advantage of imprisonment. In the intermediate case, younger offenders should be subject to prison followed by parole, while older offenders should be subject to probation. Thus, there would be a greater tendency to use prison for younger offenders who have relatively high discount rates and out-of-prison supervision for older offenders who have relatively low discount rates.

Appendix

Proof of Proposition 1: Let D_i be the present value of the disutility generated by sanction i in a given sentence, where the sentence can include multiple terms of sanction i. Then the deterrence constraint requires that $D_1 + D_2 = k$. If C is the resulting present value of the costs borne by the state, then because costs and disutilities have the same discount factors when $r = \rho$, $C = (c_1/\theta_1)D_1 + (c_2/\theta_2)D_2$. Solving the deterrence constraint for $D_2 = k - D_1$ and substituting into the expression for cost, we obtain

$$C = k(c_2/\theta_2) + D_1[(c_1/\theta_1) - (c_2/\theta_2)]. \tag{A1}$$

If $c_1/\theta_1 = c_2/\theta_2$, every sentence meeting the deterrence constraint generates the same cost, $k(c_2/\theta_2)$, so we obtain part (b).

If $c_1/\theta_1 < c_2/\theta_2$, cost decreases with D_1, so any sentence satisfying the deterrence constraint with $D_1 = k$ and $D_2 = 0$ is optimal. A sentence consisting of a prison term of length $s_1 = (1/r)ln[\theta_1/(\theta_1 - rk)]$ starting at $t = 0$ followed by an infinite term of the null sanction generates $D_1 = k$ and is therefore optimal. Part (a) follows.

If $c_1/\theta_1 > c_2/\theta_2$, cost increases with D_1, so if the state can satisfy the deterrence constraint without prison, that is, if $k \leq \theta_2/r$, then any sentence satisfying the deterrence constraint with $D_1 = 0$ and $D_2 = k$ is optimal. If $k < \theta_2/r$, a sentence consisting of a probation term of length $s_2 = (1/r)ln[\theta_2/(\theta_2 - rk)]$ starting at $t = 0$ followed by an infinite term of the null sanction generates $D_2 = k$ and is therefore optimal. Part (c)(i) follows. If $k = \theta_2/r$, an infinite probation term starting at $t = 0$ generates $D_2 = k$, and is therefore optimal. Part (c)(ii) follows.

If $c_1/\theta_1 > c_2/\theta_2$ and $k > \theta_2/r$, then an optimal sentence employs the minimum D_1 necessary to satisfy the deterrence constraint. It is straightforward to see that in this case an optimal sentence does not use the null sanction. For if a sentence employed the null sanction, it would be possible to (i) replace part of a term of the null sanction with a term of out-of-prison supervision and (ii) shorten a prison term in such a way that the present value of disutility generated by the sentence does not change, but cost drops. Since, for $r > 0$,

$$\int_0^\infty e^{-rt}dt = 1/r, \tag{A2}$$

for any sentence not using the null sanction, it must be that $(D_1/\theta_1) + (D_2/\theta_2) = 1/r$. Combining this equation with the deterrence constraint implies that for any optimal sentence,

$$D_1 = (\theta_1/r)(rk - \theta_2)/(\theta_1 - \theta_2) \tag{A3}$$

and

$$D_2 = (\theta_2/r)(\theta_1 - rk)/(\theta_1 - \theta_2). \tag{A4}$$

Any sentence with these values of D_1 and D_2 will generate the same present value of cost, and therefore be optimal. Both (i) a sentence with a prison term of length $s_1 = (1/r)ln[(\theta_1 - \theta_2)/(\theta_1 - rk)]$ starting at $t = 0$, followed by an infinite parole term, and (ii) a sentence with a parole term of length $s_2 = (1/r)ln[(\theta_2 - \theta_1)/(\theta_2 - rk)]$ starting at $t = 0$, followed by an infinite prison term, generate the required values of D_1 and D_2, and are therefore optimal. This establishes part (c)(iii). □

Proof of Proposition 2: We will demonstrate here that if $r > \rho > 0$, then any optimal sentence that includes multiple sanctions uses them in order of decreasing severity. Proposition 2 follows immediately from this claim.

As a preliminary matter, we introduce the concept of the *duration* of a continuous flow of costs or utility. Let $\phi(t)$ be the level of a continuous flow per unit time. The duration of flow $\phi(t)$ is defined as

$$\tau(\phi, x) = \left(\int_0^\infty t\phi(t)e^{-xt}dt\right)/\left(\int_0^\infty \phi(t)e^{-xt}dt\right). \tag{A5}$$

The duration can be interpreted as a weighted average of the time at which flows of costs or utility occur, where the weight at time t is proportional to the present value of the flow at t. Additionally, let $PV(\phi, x)$ denote the present value of flow $\phi(t)$ as of time $t = 0$ using discount rate $x > 0$. The duration of $\phi(t)$ can be expressed in terms of this present value as

$$\tau(\phi, x) = -\partial ln(PV(\phi, x))/\partial x. \quad \text{(A6)}$$

We now prove by contradiction that if $r > \rho > 0$, then any optimal sentence that includes multiple sanctions uses those sanctions in order of decreasing severity. Specifically, we will show that if sentence $\sigma(t)$ is optimal and, during a finite period of time $[a, b]$, $\sigma(t)$ uses two sanctions (chosen from among prison, out-of-prison supervision, and the null sanction) in order of *increasing* severity, then we can construct another sentence, $s(t)$, that reverses the order of the sanctions during the period $[a, b]$ and generates the same present value of disutility as $\sigma(t)$ at lower cost. Since the terms of the sentence outside of the period $[a, b]$ do not affect the analysis, we can assume without loss of generality that this period is $[0, T]$ for some $T > 0$ and that for all $t > T$, $\sigma(t) = s(t) = 0$. Call the less severe sanction l for low severity, and the more severe sanction h for high severity, where $\theta_l < \theta_h$. Let $L \in (0, T)$ be the length of time during which the less severe sanction is used. Thus, $\sigma(t) = \theta_l$ if $t \in [0, L]$, and $\sigma(t) = \theta_h$ if $t \in (L, T]$.[29]

Now consider an alternative sentence, $s(t)$, that is identical to the original sentence, $\sigma(t)$, except that for some $H \in (0, T)$, $s(t) = \theta_h$ if $t \in [0, H]$, and $s(t) = \theta_l$ if $t \in (H, T]$. Let H be chosen so that the present value of disutility is the same under $\sigma(t)$ and $s(t)$. It is straightforward to see that such an H exists (if $H = T - L$, the present value of disutility would be higher under $s(t)$, while if $H = 0$, it would be lower).

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[29] It is immaterial whether $\sigma(L)$ equals θ_l or θ_h. We will not comment again when similar points arise below.

Next observe that if $r = \rho > 0$, the weight on θ_i in the expression for the present value of disutility for any sentence will equal the weight on c_i in the corresponding expression for the sentence's cost. Therefore, since $\sigma(t)$ and $s(t)$ generate the same present value of disutility, $\sigma(t)$ and $s(t)$ must also generate the same present value of costs if $r = \rho > 0$:

$$PV(c(\sigma), r) = PV(c(s), r). \tag{A7}$$

We next decompose each of the flows of costs $c(\sigma(t))$ and $c(s(t))$ into two parts: one flow common to both sentences and a second flow that has a greater duration for sentence $\sigma(t)$. We then use this decomposition to prove that $PV(c(\sigma), \rho) > PV(c(s), \rho)$, implying that sentence $\sigma(t)$ is not optimal, the desired contradiction.

Define $v(t)$, $w(t)$, and $z(t)$ as follows:

$$v(t) = c_l \ \forall\ t \in [0, T], \text{ and } v(t) = 0\ \forall\ t \notin [0, T]; \tag{A8}$$

$$w(t) = c_h - c_l > 0\ \forall\ t \in (L, T], \text{ and } w(t) = 0\ \forall\ t \notin (L, T]; \tag{A9}$$

and

$$z(t) = c_h - c_l > 0\ \forall\ t \in [0, H], \text{ and } z(t) = 0\ \forall\ t \notin [0, H]. \tag{A10}$$

The definitions of $v(t)$, $w(t)$, and $z(t)$ ensure that $\forall\ t \geq 0$,

$$c(\sigma(t)) = v(t) + w(t) \tag{A11}$$

and

$$c(s(t)) = v(t) + z(t). \tag{A12}$$

The flow $z(t)$ can be converted to the flow $w(t)$ by a simple transformation: Starting with $z(t)$, increase by L the time at which positive flow begins (from $t = 0$ to $t = L$) and increase by $T - H$ the time at which positive flow ends (from $t = H$ to $t = T$). Because duration is the weighted average time at which costs are incurred, eliminating some time at the beginning of a constant flow or adding some time at its end must raise duration. Since the transformation from $z(t)$ to $w(t)$ effects both of these adjustments, it follows that:

$$\tau(w, x) > \tau(z, x) \; \forall \; x \geq 0. \tag{A13}$$

We now show that (A13) implies that $PV(c(\sigma), \rho) > PV(c(s), \rho)$. Using (A6) and $r > \rho$,

$$ln(PV(w, r)) - ln(PV(w, \rho)) = -\int_{\rho}^{r} \tau(w, x)dx \tag{A14}$$

and

$$ln(PV(z, r)) - ln(PV(z, \rho)) = -\int_{\rho}^{r} \tau(z, x)dx. \tag{A15}$$

Substituting (A11) and (A12) into (A7) yields

$$PV(w, r) = PV(z, r). \tag{A16}$$

Since by (A13), $\tau(w, x) > \tau(z, x)$ for all $x \in [\rho, r]$,

$$\int_{\rho}^{r} \tau(w, x)dx > \int_{\rho}^{r} \tau(z, x)dx. \tag{A17}$$

Given the preceding derivations, we obtain

$$\begin{aligned} ln(PV(w, \rho)) &= ln(PV(w, r)) + \int_{\rho}^{r} \tau(w, x)dx > ln(PV(w, r)) \\ &+ \int_{\rho}^{r} \tau(z, x)dx = ln(PV(z, r)) + \int_{\rho}^{r} \tau(z, x)dx = ln(PV(z, \rho)). \end{aligned} \tag{A18}$$

The first equality in (A18) follows from (A14), the inequality from (A17), the second equality from (A16), and the last equality from (A15). By (A18),

$$PV(w, \rho) > PV(z, \rho), \tag{A19}$$

so

$$PV(v, \rho) + PV(w, \rho) > PV(v, \rho) + PV(z, \rho), \tag{A20}$$

and by (A11) and (A12),

$$PV(C(\sigma), \rho) > PV(C(s), \rho). \tag{A21}$$

Since sentence $s(t)$ satisfies the deterrence constraint, (A21) contradicts the assumption that sentence $\sigma(t)$ is optimal. Thus, if $r > \rho > 0$, any sanctions used in an optimal sentence must appear in order of decreasing severity. □

Before proceeding to the proof of Proposition 3, we state and prove three lemmas that will be used in the proof.

Lemma 1: If $r > \rho > 0$, the final term of any optimal sentence must use the null sanction.

Proof of Lemma 1: Assume $r > \rho > 0$. We suppose that an optimal sentence $\sigma(t)$ uses imprisonment or out-of-prison supervision in its final term, and then derive a contradiction by constructing a sentence $s(t)$ that generates the same disutility as $\sigma(t)$ at lower cost.

Assume that the final term of $\sigma(t)$ starts at $t = S \geq 0$. Let f indicate the final sanction of sentence $\sigma(t)$, so that $\theta_f \in \{\theta_1, \theta_2\}$ is the disutility per unit time produced by f and $c(\theta_f) = c_f \in \{c_1, c_2\}$ is the corresponding cost per unit time. Let $s(t)$ differ from $\sigma(t)$ by shifting all sanctions in sentence $\sigma(t)$ forward (into the future) by a short period of time $\tau > 0$, adding a brief term of prison from $t = 0$ to $t = \tau$, and switching to the null sanction in perpetuity at $t = T \geq S + \tau$. In other words, $s(t)$ equals θ_1 for $t \in [0, \tau]$; $\sigma(t - \tau)$ for $t \in (\tau, T]$; and θ_3 for $t > T$.

Let k be the present value of the disutility generated by sentence $\sigma(t)$ and C be the present value of the cost of $\sigma(t)$. Define Δk as the present value of the disutility of sentence $s(t)$ less the present value of the disutility of sentence $\sigma(t)$; and define ΔC to be the corresponding expression for the difference in costs. We will choose T in sentence $s(t)$ to ensure that $\Delta k = 0$ and then show that $\Delta C < 0$, implying that $\sigma(t)$ cannot be optimal.

The present value of disutility generated by $s(t)$ is

$$(\theta_1/r)(1 - e^{-r\tau}) + ke^{-r\tau} - (\theta_f/r)e^{-rT}. \tag{A22}$$

The first term in (A22) is the disutility due to the initial prison term of $s(t)$. The second term is the disutility that would be generated by imposing sentence $\sigma(t)$ after a delay of τ, and the third term is the disutility lost because $s(t)$ switches from sanction f to the null sanction at time $t = T$, rather than continuing with sanction f, as does sentence $\sigma(t)$.

Therefore, for Δk to equal zero, (A22) must equal k, which can be expressed as

$$(\theta_I/r)(1 - e^{-r\tau}) + k(e^{-r\tau} - 1) - (\theta_f/r)e^{-rT} = 0. \tag{A23}$$

Solving (A23) for e^{-rT} and taking the natural logarithm of both sides yields

$$T = (1/r)ln\{\theta_f/[(\theta_I - rk)(1 - e^{-r\tau})]\}. \tag{A24}$$

Note that because $\theta_f/(\theta_I - rk)$ is finite and positive, T increases without bound as τ goes to zero, and we can always choose τ sufficiently low so that (A24) yields $T \geq S + \tau$, as assumed above. By (A24),

$$e^{-\rho T} = \{(\theta_I - rk)(1 - e^{-r\tau})/\theta_f\}^{\rho/r}. \tag{A25}$$

Analogously to (A22), the present value of costs under $s(t)$ is

$$(c_I/\rho)(1 - e^{-\rho\tau}) + Ce^{-\rho\tau} - (c_f/\rho)e^{-\rho T}. \tag{A26}$$

Therefore, for $\sigma(t)$ to be optimal, it must be that $\Delta C \geq 0$, that is,

$$(c_I/\rho)(1 - e^{-\rho\tau}) + C(e^{-\rho\tau} - 1) - (c_f/\rho)e^{-\rho T} \geq 0. \tag{A27}$$

Rearranging (A27) so that $e^{-\rho T}$ is on the right-hand side and then employing (A25) yields

$$[(c_I - \rho C)/c_f](1 - e^{-\rho\tau}) \geq \{(\theta_I - rk)(1 - e^{-r\tau})/\theta_f\}^{\rho/r}. \tag{A28}$$

Note that $c_I - \rho C > 0$; this condition is equivalent to $C < c_I/\rho$, which must hold unless $\sigma(t)$ is a perpetual prison term, which it cannot be if $k < \theta_I/r$. Thus, (A28) can be rewritten as

$$(1 - e^{-\rho\tau})/(1 - e^{-r\tau})^{\rho/r} \geq \{(\theta_I - rk)/\theta_f\}^{\rho/r}[c_f/(c_I - \rho C)]. \tag{A29}$$

Since $\theta_I - rk > 0$ (implied by the assumption that $k < \theta_I/r$) and $c_I - \rho C > 0$, the right-hand side of (A29) is a positive constant. By applying L'Hopital's rule, it can be shown that the left-hand side of (A29) goes to 0 as τ goes to 0. It follows that for sufficiently low values of τ, inequality (A29) cannot hold and $\sigma(t)$ cannot be optimal. We have thus demonstrated the desired contradiction. □

Lemma 2: Assume $r > \rho > 0$ and let $\sigma(t)$ be an optimal sentence. Then:

(a) $\sigma(t)$ consists of a prison term of length $s_1 \geq 0$, followed by a term of out-of-prison supervision of length $s_2 \geq 0$ (where at most one of s_1 and s_2 can be zero), followed by an infinite term of the null sanction;

(b) $\sigma(t)$ generates disutility of

$$D(s_1, s_2) = (\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)[e^{-rs_1} - e^{-r(s_1 + s_2)}]; \tag{A30}$$

(c) $\sigma(t)$ generates cost of

$$C(s_1, s_2) = (c_1/\rho)(1 - e^{-\rho s_1}) + (c_2/\rho)[e^{-\rho s_1} - e^{-\rho(s_1 + s_2)}]; \tag{A31}$$

and

(d) letting

$$L(s_1, s_2, \lambda, \mu_1, \mu_2) = C(s_1, s_2) + \lambda(k - D(s_1, s_2)) - \mu_1 s_1 - \mu_2 s_2, \tag{A32}$$

there exist Kuhn-Tucker multipliers λ, $\mu_1 \geq 0$, and $\mu_2 \geq 0$, such that $\sigma(t)$ satisfies the following Kuhn-Tucker conditions:

$$\partial L/\partial s_i = 0 \ \forall i \in \{1, 2\}; \tag{A33}$$

$$\lambda(k - D(s_1, s_2)) = 0; \tag{A34}$$

and

$$\mu_i s_i = 0 \ \forall i \in \{1, 2\}. \tag{A35}$$

Proof of Lemma 2: Part (a) follows immediately from Lemma 1 and the proof of Proposition 2.

Thus, the disutility of sentence $\sigma(t)$ is

$$D(s_1, s_2) = \theta_1 \int_0^{s_1} e^{-rt}dt + \theta_2 \int_{s_1}^{s_1+s_2} e^{-rt}dt, \tag{A36}$$

which can be written as (A30). The cost of sentence $\sigma(t)$ is an analogous expression with ρ substituted for r and c_i substituted for θ_i, given by (A31). This confirms parts (b) and (c) of the lemma.

Given these results, the state's problem described in Section 2 can be expressed as the following constrained optimization problem, which will be referred to as the *optimal sentencing problem*:

$$\min_{s_1, s_2} C(s_1, s_2) \tag{A37}$$

subject to

$$k - D(s_1, s_2) = 0 \tag{A38}$$

and

$$-s_i \leq 0 \ \forall i \in \{1, 2\}. \tag{A39}$$

The Lagrangean function corresponding to the optimal sentencing problem can be written as (A32). By the Kuhn-Tucker Theorem, if a technical condition called the constraint qualification holds, then at any solution to the optimal sentencing problem, there must be a number λ and non-negative numbers s_1, s_2, μ_1, and μ_2 such that (A33) through (A35) also hold. We will refer to (A33) through (A35) as the Kuhn-Tucker conditions for the optimal sentencing problem. The constraint qualification condition is satisfied for this problem, though for brevity we will not demonstrate this here. Thus, we have obtained part (d) of the lemma. □

Lemma 3: Either (a) $(c_1 - c_2)/(\theta_1 - \theta_2) < c_1/\theta_1 < c_2/\theta_2$; or (b) $c_2/\theta_2 < c_1/\theta_1 < (c_1 - c_2)/(\theta_1 - \theta_2)$; or (c) $c_1/\theta_1 = c_2/\theta_2 = (c_1 - c_2)/(\theta_1 - \theta_2)$.

Proof of Lemma 3: (a) Suppose that $c_1/\theta_1 < c_2/\theta_2$. From this expression it can be established that $(c_1 - c_2)/(\theta_1 - \theta_2) < c_1/\theta_1$ through the following steps. Multiply both sides by

$\theta_1\theta_2$; subtract $c_1\theta_1$ from both sides; factor out c_1 on the left-hand side and θ_1 on the right-hand side; multiply both sides by $-1/[(\theta_1 - \theta_2)\theta_1]$. This establishes part (a).

(b) Now suppose that $c_2/\theta_2 < c_1/\theta_1$. By a procedure analogous to that employed in part (a) it can be shown that $c_1/\theta_1 < (c_1 - c_2)/(\theta_1 - \theta_2)$, thereby establishing part (b). The only difference is that in the third step θ_1 should be factored out on the left-hand side and c_1 on the right-hand side.

(c) Finally, suppose that $c_1/\theta_1 = c_2/\theta_2$. Using the same procedure as in part (a) it can be shown that $c_1/\theta_1 = (c_1 - c_2)/(\theta_1 - \theta_2)$, thereby demonstrating part (c). □

Proof of Proposition 3: Lemma 2 provides the Lagrangean function for the optimal sentencing problem and the Kuhn-Tucker conditions that any solution to this problem must satisfy. We will rewrite the Kuhn-Tucker conditions in terms of the parameters of the optimal sentencing problem, and then solve for the term lengths s_1 and s_2. Last, we will argue that the Kuhn-Tucker conditions are sufficient as well as necessary for optimality.

By Lemma 2, the Lagrangean for the optimal sentencing problem can be expressed as:

$$\begin{aligned} L(s_1, s_2, \lambda, \mu_1, \mu_2) &= (c_1/\rho)(1 - e^{-\rho s_1}) + (c_2/\rho)[e^{-\rho s_1} - e^{-\rho(s_1+s_2)}] \\ &+ \lambda(k - (\theta_1/r)(1 - e^{-rs_1}) - (\theta_2/r)[e^{-rs_1} - e^{-r(s_1+s_2)}]) - \mu_1 s_1 - \mu_2 s_2. \end{aligned} \tag{A40}$$

Condition (A33) requires that $\partial L/\partial s_1 = 0$ and $\partial L/\partial s_2 = 0$, which can be written, respectively, as

$$\mu_1 = (c_1 - c_2)e^{-\rho s_1} + c_2 e^{-\rho(s_1+s_2)} + \lambda((\theta_2 - \theta_1)e^{-rs_1} - \theta_2 e^{-r(s_1+s_2)}) \tag{A41}$$

and

$$\mu_2 = c_2 e^{-\rho(s_1+s_2)} - \lambda\theta_2 e^{-r(s_1+s_2)}. \tag{A42}$$

Condition (A34) requires that

$$\lambda(k - \{(\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)[e^{-rs_1} - e^{-r(s_1+s_2)}]\}) = 0. \tag{A43}$$

Condition (A35) requires that

$$\mu_1 s_1 = 0 \tag{A44}$$

and

$$\mu_2 s_2 = 0. \tag{A45}$$

To derive the optimal term lengths for s_1 and s_2, it will be convenient to consider four mutually exclusive and collectively exhaustive cases. In each case, we will assume that the pair of term lengths (s_1, s_2) satisfies the Kuhn-Tucker conditions, (A41)-(A45).

Case 1: $s_1 > 0, s_2 = 0$

Since $s_1 > 0$, (A44) requires that $\mu_1 = 0$. Solving (A41) for λ, with $s_2 = \mu_1 = 0$, yields

$$\lambda = (c_1/\theta_1)e^{(r-\rho)s_1} > 0. \tag{A46}$$

Then solving (A43) for s_1, with $s_2 = 0$ and λ given by (A46), yields

$$s_1 = (1/r)ln[\theta_1/(\theta_1 - rk)] > 0. \tag{A47}$$

Since we have been assuming that $k < \theta_1/r$, the argument of the natural logarithm function in (A47) exceeds unity and hence the preceding expression for s_1 is well-defined.

As noted in the proof of Lemma 2, the Kuhn-Tucker Theorem requires that the multipliers μ_1 and μ_2 be non-negative. Since in the present case $s_2 = 0$, (A45) implies that $\mu_2 \geq 0$. By substituting (A46) into (A42) and using $s_2 = 0$, the condition that $\mu_2 \geq 0$ can be expressed as

$$\mu_2 = c_2 e^{-\rho s_1} - (c_1/\theta_1)e^{-\rho s_1}\theta_2 \geq 0, \tag{A48}$$

which is equivalent to

$$c_1/\theta_1 \leq c_2/\theta_2. \tag{A49}$$

Hence, case 1 is possible only when $c_1/\theta_1 \leq c_2/\theta_2$, and then for $k < \theta_1/r$, s_1 is given by (A47) and $s_2 = 0$.

Case 2: $s_1 = 0, s_2 > 0$

Since $s_2 > 0$, (A45) requires that $\mu_2 = 0$. Then solving (A42) for λ, with $s_1 = 0$, yields

$$\lambda = (c_2/\theta_2)e^{(r-\rho)s_2} > 0. \tag{A50}$$

Solving (A43) for s_2, using (A50) and $s_1 = 0$, results in

$$s_2 = (1/r)ln[\theta_2/(\theta_2 - rk)] > 0. \tag{A51}$$

To guarantee that this expression for s_2 is well-defined, we assume that $k < \theta_2/r$. Below, we show that k must satisfy an even more restrictive condition for the present case to arise.

Since in the present case $s_1 = 0$, (A44) implies that $\mu_1 \geq 0$. By substituting (A50) into (A41) and using $s_1 = 0$, the condition that $\mu_1 \geq 0$ can be expressed as

$$c_1 - c_2 + c_2e^{-\rho s_2} + (c_2/\theta_2)e^{(r-\rho)s_2}(\theta_2 - \theta_1 - \theta_2e^{-rs_2}) \geq 0. \tag{A52}$$

Multiply both sides of (A51) by $-r$ and apply the exponential function to obtain

$$e^{-rs_2} = (\theta_2 - rk)/\theta_2. \tag{A53}$$

Raising both sides of (A53) by the power of ρ/r yields

$$e^{-\rho s_2} = [(\theta_2 - rk)/\theta_2]^{\rho/r}. \tag{A54}$$

Now insert (A53) and (A54) into (A52) and rearrange terms to obtain

$$c_1 - c_2 \geq (c_2/\theta_2)e^{(r-\rho)s_2}(\theta_1 - rk) - c_2[(\theta_2 - rk)/\theta_2]^{\rho/r}. \tag{A55}$$

Raise both sides of (A53) by the power of $(\rho - r)/r$ and substitute the resulting right-hand side for $e^{(r-\rho)s_2}$ in (A55); after factoring out the term $[(\theta_2 - rk)/\theta_2]^{\rho/r}$ on the right-hand side of (A55), the result is

$$c_1 - c_2 \geq [(\theta_2 - rk)/\theta_2]^{\rho/r}[\{(c_2/\theta_2)(\theta_1 - rk)/[(\theta_2 - rk)/\theta_2]\} - c_2]. \tag{A56}$$

After multiplying both sides of (A56) by θ_2/c_2 and then rewriting $\{(\theta_1 - rk)/[(\theta_2 - rk)/\theta_2]\} - \theta_2$ as $(\theta_1 - \theta_2)/[(\theta_2 - rk)/\theta_2]$, (A56) can be expressed as

$$[(c_1 - c_2)/(\theta_1 - \theta_2)](\theta_2/c_2) \geq [(\theta_2 - rk)/\theta_2]^{(\rho - r)/r}. \tag{A57}$$

Raise both sides of (A57) by the power of $r/(r-\rho)$; multiply each side by $(\theta_2 - rk)\{[(\theta_1 - \theta_2)/(c_1 - c_2)](c_2/\theta_2)\}^{r/(r-\rho)}$; and then solve for k to obtain

$$k \leq \kappa, \tag{A58}$$

where

$$\kappa = (\theta_2/r) - (\theta_2/r)\{(c_2/\theta_2)[(\theta_1 - \theta_2)/(c_1 - c_2)]\}^{r/(r-\rho)}. \tag{A59}$$

Since $k > 0$, in order for condition (A58) to possibly hold, it must be that $\kappa > 0$. It is straightforward to show from (A59) that κ is positive if and only if $(c_1 - c_2)/(\theta_1 - \theta_2) > c_2/\theta_2$. By part (b) of Lemma 3, this implies that $c_2/\theta_2 < c_1/\theta_1$. Moreover, if $(c_1 - c_2)/(\theta_1 - \theta_2) > c_2/\theta_2$, the term in braces in (A59) is less than 1, implying that $\kappa < \theta_2/r$.

In sum, Case 2 is possible only when $c_1/\theta_1 > c_2/\theta_2$ and $k \leq \kappa$, where $\kappa < \theta_2/r$ is given by (A59), in which case $s_1 = 0$ and s_2 is given by (A51).

Case 3: $s_1 > 0$, $s_2 > 0$

Since $s_1 > 0$ and $s_2 > 0$, (A44) and (A45) require that $\mu_1 = \mu_2 = 0$. Solving (A42) for λ yields

$$\lambda = (c_2/\theta_2)e^{(r-\rho)(s_1 + s_2)} > 0. \tag{A60}$$

After substituting λ from (A60) into (A41) and simplifying, (A41) can be written as

$$[(c_1 - c_2)/(\theta_1 - \theta_2)](\theta_2/c_2) = e^{(r-\rho)s_2}. \tag{A61}$$

Observe that

$$e^{-rs_2} = (e^{(r-\rho)s_2})^{-r/(r-\rho)} = \{(c_2/\theta_2)[(\theta_1 - \theta_2)/(c_1 - c_2)]\}^{r/(r-\rho)} = (\theta_2 - r\kappa)/\theta_2, \tag{A62}$$

where the second equality follows from (A61) and the third equality from (A59). Taking the natural logarithms of the first and last terms in (A62) and solving for s_2 yields

$$s_2 = (1/r)ln[\theta_2/(\theta_2 - r\kappa)]. \tag{A63}$$

This expression is well-defined and positive provided that $\theta_2/(\theta_2 - r\kappa) > 1$ or, equivalently, given (A59), if $(c_1 - c_2)/(\theta_1 - \theta_2) > c_2/\theta_2$. By Lemma 3, this implies that $c_2/\theta_2 < c_1/\theta_1$.

Since $\lambda > 0$ (see (A60)), (A43) implies that

$$(\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)[e^{-rs_1} - e^{-r(s_1 + s_2)}] = k. \tag{A64}$$

If one factors out e^{-rs_1} from the second term and uses (A62) to substitute for e^{-rs_2}, this expression can be rewritten as

$$(\theta_1/r)(1 - e^{-rs_1}) + (\theta_2/r)e^{-rs_1}[r\kappa/\theta_2] = k. \tag{A65}$$

Solving (A65) for e^{-rs_1} and taking natural logarithms of both sides yields, after further manipulation,

$$s_1 = (1/r)ln[(\theta_1 - r\kappa)/(\theta_1 - rk)]. \tag{A66}$$

This expression is well-defined and positive provided that $(\theta_1 - r\kappa)/(\theta_1 - rk) > 1$ or, equivalently, $k > \kappa$.

Thus, Case 3 is possible only when $c_1/\theta_1 > c_2/\theta_2$ and $k > \kappa$, where κ is given by (A59), in which case s_1 is given by (A66) and s_2 by (A63).

Case 4: $s_1 = 0$, $s_2 = 0$

If $s_1 = s_2 = 0$, the sentence obviously cannot generate any disutility and therefore cannot be a solution to the optimal sentencing problem.

Together, the proof of Proposition 2 and Lemma 1 showed that when $r > \rho > 0$, an optimal sentence consists of a term of prison, followed by a term of out-of-prison supervision, followed by an infinite term of the null sanction (with one of the first two terms possibly of zero length). Lemma 2 showed that the ordered pair of term lengths (s_1, s_2) for an optimal sentence must satisfy the Kuhn-Tucker conditions. It can be demonstrated that if $r > \rho > 0$, a solution to the optimal sentencing problem exists.[30] Thus, since the Kuhn-Tucker conditions have only a single solution in each of Cases 1 through 3, these conditions are necessary and sufficient for optimality; and the term lengths we derived above for each case are uniquely optimal.

If $c_1/\theta_1 \leq c_2/\theta_2$, only Case 1 ($s_1 > 0$, $s_2 = 0$) is possible, and the unique solution to the optimal sentencing problem is a prison term of length (A47) starting at $t = 0$. Obviously, this result holds for any target level of deterrence $k \in (0, \theta_1/r)$. We have thus demonstrated (a).

If $c_1/\theta_1 > c_2/\theta_2$ and $k \leq \kappa$, only Case 2 ($s_1 = 0$, $s_2 > 0$) is possible, and the unique solution to the optimal sentencing problem consists of a probation term of length (A51) starting at $t = 0$. We have thus demonstrated (b)(i).

If $c_1/\theta_1 > c_2/\theta_2$ and $k > \kappa$, only Case 3 ($s_1 > 0$, $s_2 > 0$) is possible, and the unique solution to the optimal sentencing problem consists of a prison term of length (A66) starting at $t = 0$, followed immediately by a parole term of length (A63). We have thus demonstrated (b)(ii). □

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[30] One can formally prove the existence of a solution in two steps: (a) showing that optimal term lengths (s_1, s_2) have finite upper bounds, so that the optimal sentencing problem is equivalent to a minimization over a closed and bounded set of term lengths; and (b) then applying the Weierstrass Theorem, which states that any continuous function has a minimum on a closed and bounded set.

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