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By

John William Hatfield, Fuhito Kojima, Yusuke Narita

Stanford Institute for Economic Policy Research
Stanford University
Stanford, CA 94305
(650) 725-1874

Many-to-Many Matching with Max-Min Preferences☆

John William Hatfield[a], Fuhito Kojima[b,*], Yusuke Narita[c]

[a] *Graduate School of Business, Stanford University*
[b] *Department of Economics, Stanford University*
[c] *Department of Economics, MIT*

Abstract

We consider the many-to-many two-sided matching problem under a stringent domain restriction on preferences called the max-min criterion. We show that, even under this restriction, there is no stable mechanism that is weakly Pareto efficient, strategy-proof, or monotonic (i.e. respects improvements) for agents on one side of the market. These results imply in particular that three of the main results of [4] are *in*correct.

Keywords: Many-to-Many Two-Sided Matching; Stability; Pareto Efficiency; Monotonicity; Strategy-proofness; Max-Min Preferences

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*Corresponding author

Email addresses: `hatfield_john@gsb.stanford.edu` (John William Hatfield), `fkojima@stanford.edu` (Fuhito Kojima), `narita@mit.edu` (Yusuke Narita)

1. Introduction

The celebrated deferred acceptance algorithm of [8] not only finds a stable matching, but also is weakly Pareto efficient and strategy-proof for agents on the proposing side of the market, so long as each of those agents has unit demand [12]. Moreover, it is monotonic in the sense that an agent is weakly better off if she becomes more preferred by others [6]. Given these desirable features, the deferred acceptance algorithm is used in practical matching problems such as public school choice [1, 2, 3].

By sharp contrast, it is well-known that, in the more general many-to-many setting, no stable mechanism is weakly Pareto efficient, strategy-proof, or monotonic.[1] Given that these properties may be important for the proper functioning of many-to-many markets in numerous contexts, the lack of these properties may make stable matching mechanisms less desirable for practical application in many-to-many matching markets. A natural question, then, is whether there is any restriction on preferences that enables a stable mechanism to satisfy these properties for agents with multi-unit demand on one side of the market.

This paper considers the many-to-many matching problem under a stringent domain restriction on preferences called the "max-min criterion", introduced by [4]. It is shown that, even under the restriction, there is no stable mechanism that, for agents on one side of the market, is either weakly Pareto efficient, strategy-proof, or monotonic. In particular, our result implies that three of the main results (Theorems 5, 6, and 7) of [4] are *in*correct.

2. Model

There is a finite set R of **row-players** and a finite set C of **column-players**.[2] Each $c \in C$ has a strict preference relation $\succ_c$ over R and the outside option denoted by $\varnothing$ and its **quota** q_c. The preference profile of all column-players is denoted by $\succ_C \equiv (\succ_c)_{c \in C}$. The weak preference relation associated with $\succ_c$ is denoted by $\succsim_c$ and so we write $r_1 \succsim_c r_2$ (where $r_1, r_2 \in R$) if either $r_1 \succ_c r_2$ or $r_1 = r_2$. Corresponding notation is also used for the row-players. We denote the quota of row-player r by p_r. A preference profile of all players is denoted by $\succ \equiv (\succ_R, \succ_C)$.

We extend the preferences (over individuals) to those over subsets of agents on the other side of the market. Following [4], we say that the preference relation of $r \in R$ satisfies the **max-min criterion** if the following condition is met: For any $C_1, C_2 \subseteq C$ with $|C_1| \leq p_r$ and $|C_2| \leq p_r$, if (1) $|C_1| \geq |C_2|$ and r strictly prefers the *least* preferred column-player

[1] See, for instance, [5] who use an example of [11] to exhibit this fact.

[2] For example, row- and column-players may correspond to workers and firms.

in C_1 to the least preferred column-player in C_2, or (2) $C_1 = C_2$, then $C_1 \succsim_r C_2$.[3] The preference relation of c satisfies the max-min criterion if the corresponding condition is met. Throughout the paper, we assume that the preference relation of every player satisfies the max-min criterion.

2.1. Matching Mechanisms and Their Properties

A **matching** is a vector $\mu = (\mu(r))_{r \in R}$ that assigns each r a set of at most p_r column-players $\mu(r) \subseteq C$, and each $c \in C$ is also assigned at most q_c row-players. We denote by $\mu(c) \equiv \{r \in R | c \in \mu(r)\}$ the set of row players who are assigned to c.

A matching μ is **individually rational** if $j \succsim_i \varnothing$ for every $i \in C \cup R$ and every $j \in \mu(i)$.[4] A matching μ is **blocked** by $(r, c) \in R \times C$ if (1) $c \succ_r \varnothing$ and $r \succ_c \varnothing$, (2) $|\mu(r)| < p_r$ or $c \succ_r c'$ for some $c' \in \mu(r)$, and (3) $|\mu(c)| < q_c$ or $r \succ_c r'$ for some $r' \in \mu(c)$. A matching μ is **stable** if it is individually rational and it is not blocked; it is well-known that a stable matching always exists [12]. A matching μ is **weakly row-efficient** if there exists no individually rational matching μ' such that $\mu'(r) \succ_r \mu(r)$ for all $r \in R$.[5] A weakly column-efficient matching is also defined in the same way.

Given the player sets R and C, a **mechanism** is a function from the set of (reported) preference profiles to the set of matchings. A mechanism is **stable** if the outcome of that mechanism is a stable matching for every preference profile. A mechanism is **weakly row(column)-efficient** if the outcome of that mechanism is a weakly row(column)-efficient matching for every preference profile. A mechanism is **row-strategy-proof** if at every preference profile, no row player can obtain a strictly better set of column-players by misreporting her preferences. A column-strategy-proof mechanism is also defined in the same fashion.

To define one more property of a mechanism, we first introduce the following concept: Preference relation $\succ'_r$ is an **improvement for** c over $\succ_r$ if

(1) For all $c_1 \in C \cup \{\varnothing\}$, if $c \succ_r c_1$, then $c \succ'_r c_1$,
(2) For all $c_1, c_2 \in (C \cup \{\varnothing\}) \setminus c$, $c_1 \succ'_r c_2$ if and only if $c_1 \succ_r c_2$,

and the capacity associated with $\succ'_r$ is equal to that with $\succ_r$. A row-player preference profile $\succ'_R$ is an improvement for c over $\succ_R$ if for every r, $\succ'_r$ is an improvement for c over $\succ_r$. We now define the following property of a mechanism: A mechanism φ is **column-monotone** if, for any preference profile $\succ$, any $c \in C$, and any row-player preference profile $\succ_R$ and

[3] Similar conditions were studied by, for instance, [13, 7, 10].

[4] Throughout the paper, we denote singleton set $\{x\}$ by x when there is no room for confusion.

[5] This property is called "row-efficiency" by [4]. Here we add "weakly" to emphasize the difference between this property and standard Pareto efficiency.

$\succ'_R$, if $\succ'_R$ is an improvement for c over $\succ_R$, then c weakly prefers $\varphi(\succ'_R, \succ_C)$ to $\varphi(\succ_R, \succ_C)$. This definition requires that the outcome of a mechanism be weakly better for a column-player if that column-player becomes more preferred by the row-players. A **row-monotone** mechanism is defined analogously. This property is first introduced by [6] as "respecting improvements" and they analyze it in a class of many-to-one matching problems.

3. Results

Consider the following example.[6] Let $R = \{r_1, r_2\}$ and $C = \{c_1, c_2\}$. Consider the following preferences:

$$\begin{aligned}
&\succ_{r_1}: c_2, c_1, \emptyset,\\
&\succ_{r_2}: c_2, c_1, \emptyset,\\
&\succ_{c_1}: r_1, r_2, \emptyset,\\
&\succ_{c_2}: r_2, r_1, \emptyset,
\end{aligned}$$

where the notational convention is that r_1 prefers c_2 most, c_1 second, and $\emptyset$ third, and so forth. (This notation is used throughout.) The quotas of the players are given by $q_{c_1} = 2$ and $q_{c_2} = p_{r_1} = p_{r_2} = 1$. Finally, let the preferences of each agent over sets of agents on the other side of the market be consistent with the max-min criterion.

Let φ be any stable mechanism. Under the preference profile $\succ \equiv (\succ_{r_1}, \succ_{r_2}, \succ_{c_1}, \succ_{c_2})$, the following matching is the unique stable matching:

$$\varphi(\succ) = \begin{pmatrix} c_1 & c_2 \\ r_1 & r_2 \end{pmatrix},$$

where this matrix notation represents the matching where c_1 is matched with r_1 while c_2 is matched with r_2. (Again, this notation is used throughout.) Stability of $\varphi(\succ)$ immediately follows from the definition. To see that $\varphi(\succ)$ is the unique stable matching, note that in any stable matching, every row player has to be matched to a column player. (If there is an unmatched row player, then there is also an unmatched column player, who in turn blocks the matching with the unmatched row player.) The only such individually rational

[6] This example is borrowed from [9].

matchings other than $\varphi(\succ)$ are

$$\begin{pmatrix} c_1 & c_2 \\ r_2 & r_1 \end{pmatrix} \text{ and } \begin{pmatrix} c_1 & c_2 \\ \{r_1, r_2\} & \emptyset \end{pmatrix},$$

but (r_2, c_2) blocks both matchings.

Now, consider a different set of preferences for agent r_2, $\succ'_{r_2}$: $c_1, c_2, \emptyset$. Note that $\succ'_{r_2}$ is an improvement for c_1 over $\succ_{r_2}$. It is easy to see that under preference profile $(\succ'_{r_2}, \succ_{-r_2})$,[7], the following matching is the unique stable matching:

$$\varphi(\succ'_{r_2}, \succ_{-r_2}) = \begin{pmatrix} c_1 & c_2 \\ r_2 & r_1 \end{pmatrix}.$$

To see the uniqueness, note that by the same reason as in the previous paragraph, in any stable matching, every row player is matched with a column player. Except for $\varphi(\succ'_{r_2}, \succ_{-r_2})$, the only such individually rational matchings are

$$\begin{pmatrix} c_1 & c_2 \\ r_1 & r_2 \end{pmatrix} \text{ and } \begin{pmatrix} c_1 & c_2 \\ \{r_1, r_2\} & \emptyset \end{pmatrix}$$

but (r_2, c_1) and (r_1, c_2) block these matchings, respectively.

For this example, first recall that at preference profile $(\succ'_{r_2}, \succ_{-r_2})$, the unique stable matching is $\varphi(\succ'_{r_2}, \succ_{-r_2})$ while both column-players strictly prefer another (unstable) matching

$$\begin{pmatrix} c_1 & c_2 \\ r_1 & r_2 \end{pmatrix}$$

to $\varphi(\succ'_{r_2}, \succ_{-r_2})$. This means that *no* stable mechanism produces a weakly column-efficient matching at $(\succ'_{r_2}, \succ_{-r_2})$. Thus the following result holds.

Theorem 1. *There is no stable mechanism that is weakly row- or column-efficient even if the preferences of the players satisfy the max-min criterion.*[8,9]

[7]Subscript $-i$ indicates $(C \cup R) \setminus i$, that is, the set of all agents except for i. For instance, $\succ_{-r_2}$ is the profile of preferences of all row- and column-players except for r_2.

[8]The preceding discussion deals only with weak column-efficiency, but the same argument can also be applied to weak row-efficiency thanks to the symmetry of the two sides of the market (row- and column-players).

[9]One can also show our Theorem 1 by adapting the example depicted in Figure 4 of [5] (which considers a related setting but allows for fractional matchings) to our setting.

Also, in the above example, even though $\succ'_{r_2}$ is an improvement for c_1 over $\succ_{r_2}$, c_1 strictly prefers $\varphi(\succ)$ (under which c_1 obtains r_1) to $\varphi(\succ'_{r_2}, \succ_{-r_2})(c_1)$ (under which c_1 obtains r_2). Thus, for any stable mechanism, φ is *not* column- or row-monotone at $\succ$, which implies the following result.

Theorem 2. *There is no stable mechanism that is row- or column-monotone even if the preferences of the players satisfy the max-min criterion.*

Finally, assume that the true preference profile is $(\succ'_{r_2}, \succ_{-r_2})$ and consider the following preference relation of c_1:

$$\succ'_{c_1}: r_1, \emptyset.$$

Then it is easy to verify that at preference profile $(\succ_{r_1}, \succ'_{r_2}, \succ'_{c_1}, \succ_{c_2})$, the unique stable matching is

$$\begin{pmatrix} c_1 & c_2 \\ r_1 & r_2 \end{pmatrix}.$$

Given that $r_1 \succ_{c_1} r_2$ and that r_2 is the only row-player matched to c_1 under $\varphi(\succ'_{r_2}, \succ_{-r_2})(c_1)$, we have that for any stable mechanism, reporting $\succ'_{c_1}$ (instead of true $\succ_{c_1}$) is a profitable deviation for c_1 at $(\succ'_{r_2}, \succ_{-r_2})$, proving the following result.

Theorem 3. *There is no stable mechanism that is row- or column-strategy-proof even if the preferences of the players satisfy the max-min criterion.*

Consequently, the above Theorems imply the following result.

Corollary 1. *All of the following claims by [4] are incorrect:*

- *If the preferences of the players satisfy the max-min criterion, then the so-called "row(column)-optimal stable" mechanism is the unique stable mechanism that is weakly row(column)-efficient (Theorem 5).*
- *If the preferences of the players satisfy the max-min criterion, then the row(column)-optimal stable mechanism is the unique stable mechanism that is row(column)-monotone (Theorem 6).*
- *If the preferences of the players satisfy the max-min criterion, then the row(column)-optimal stable mechanism is the unique stable mechanism that is row(column)-strategy-proof (Theorem 7).*

Remark. Note that the max-min criterion as defined by [4] only imposes on the preferences of an agent a partial ordering over sets of agents on the other side of the market. Nevertheless, the above example is constructed so that Corollary 1 remains valid as well for any specification of the agents' preferences consistent with the max-min criterion.

Remark. Note that in the above example, the quotas of both row players are one. Therefore, even for the many-to-*one* matching problem under the restriction of the max-min criterion (and responsiveness due to [11]), the impossibilities in Theorems 1, 2, and 3 are still true and thus the three claims by [4] do not hold.

4. Discussion

In [5], Baïou and Balinski consider a related model but allow for fractional matchings (representing, e.g., the amount of time) between players, imposing pair-specific bounds on the size of the relationship between a pair of agents (in addition to the usual quotas on the total capacity for each agent). Analogous results to our Theorems 1 to 3 hold even in that setting if agents' preferences are only required to satisfy the max-min criterion. The counterexample constructed above can easily be adapted to that setting by showing that, for both preference profiles used in our counterexample, the unique stable match in our setting corresponds to a unique stable match in the setting of [5].

However, in [5], Baïou and Balinski introduce a more stringent condition on preferences, the *generalized max-min criterion*. This criterion imposes, in the context of [4], the additional restriction on preferences that an agent is indifferent between any two allocations in which that agent is assigned the same number of partners yet does not fulfill his quota.[10] In particular, for the special case where the quota of each column-player is guaranteed to be fulfilled, the max-min criterion and generalized max-min preferences coincide. When agents have generalized max-min preferences, [5] state that the row-optimal mechanism is the unique weakly row-efficient (Theorem 1), row-monotone (Theorem 3), and row-strategy-proof (Theorem 4) mechanism. These results of [5] would imply that the conclusions of our Corollary no longer hold when the more stringent condition of generalized max-min preferences is imposed. Note that, in particular, our counterexample does not apply when the assumption of generalized max-min preferences is imposed, as under generalized max-min preferences, $\{r_1\} \not\succ_{c_1} \{r_2\}$ even though $r_1 \succ_{c_1} r_2$.[11]

[10]Note that the setting of [5] allows for fractional matchings, so this additional restriction of generalized max-min preferences is that an agent is indifferent between any two allocations in which that agent is assigned the same total number of hours while not fulfilling his quota.

[11]Recall that the quota of c_1 is 2.

4.1. Counterexample to Theorem 1 of [5]

However, Theorem 1 of [5] that the row-optimal stable mechanism is the unique weakly row-efficient mechanism is incorrect. Moreover, this lack of uniqueness carries over to our setting. In order to demonstrate our counterexample, we use terminology and notation of [5] in the remainder of this paper. Let x be a matrix indexed by row and column agents such that $x(i,j)$ represents the amount of time that row-agent i works for column-agent j. The matrix x is an allocation if all quotas (both for individual agents and pairs) are satisfied. In [5], an allocation x is said to be *row-efficient* if there is no allocation y, stable or not, satisfying $y \succ_i x$ for every row-agent i. Note that this is the same notion as the concept which [4] and our paper call "weak row-efficiency,"[12] and it is "weak" in that y only precludes x from being weakly row-efficient if *every* row agent *strictly* prefers y to x. Following [5], let χ_I denote the "row-optimal" stable mechanism, i.e., the mechanism which always selects the row-optimal stable allocation.[13]

Claim 1 (Theorem 1 of [5])**.** If the preferences of the players satisfy the generalized max-min criterion, then χ_I is the unique row-efficient stable mechanism.

This claim is incorrect. Specifically, the following counterexample shows that there exists another row-efficient stable mechanism (assuming that χ_I is row-efficient). Let $R = \{r_1, r_2, r_3\}$ and $C = \{c_1, c_2, c_3\}$. Consider the following preference profile $\succ$:

$$\begin{array}{lll} \succ_{r_1}: c_1, c_2, c_3, & \succ_{r_2}: c_2, c_1, c_3, & \succ_{r_3}: c_3, c_2, c_1, \\ \succ_{c_1}: r_2, r_1, r_3, & \succ_{c_2}: r_1, r_2, r_3, & \succ_{c_3}: r_3, r_1, r_2. \end{array}$$

Let the capacity of every agent be one (as well as the capacity of each pair). Note that the preferences of each agent are consistent with the generalized max-min criterion of [5] as each agent has a capacity of one.[14]

Under the above preference profile $\succ$, there are two stable allocations:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ r_1 & r_2 & r_3 \end{pmatrix}, \ \mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ r_2 & r_1 & r_3 \end{pmatrix},$$

[12] Recall that the setting of [5] is slightly different from that of [4] and our paper in that the former allows for both fractional allocations while the latter allows only binary allocations. However, the example that we present here works for both settings.

[13] It is well-known that a row-optimal stable allocation (i.e., a stable allocation x such that $x \succsim_i y$ for every row-agent i and for every stable allocation y) exists and is unique.

[14] The agents' preferences over fractional allocations are defined according to the generalized max-min criterion, although it does not affect the following argument.

where our matrix notation now denotes that μ is a matching where c_1 is matched with r_1, c_2 is matched with r_2, and c_3 is matched with r_3 up to their full capacity of one. μ is the row-optimal stable allocation and μ' is the column-optimal stable allocation. Note that, however, both μ and μ' are row-efficient. This is because r_3 is matched to his most preferred partner c_3 in both μ and μ', so there cannot exist any other allocation that r_3 strictly prefers.

Now, consider an allocation mechanism ϕ such that

(1) for the above preference profile $\succ$, ϕ selects μ', and
(2) ϕ selects the row-optimal stable matching for any other preference profile.

This mechanism is stable, and it is obviously different from χ_I (recall that χ_I is defined as the mechanism that produces the row-optimal stable allocation for *every* preference profile). However, ϕ is row-efficient if χ_I is row-efficient because

(1) μ' is row-efficient for preference profile $\succ$ by the above argument, and
(2) for any other preference profile, ϕ coincides with the row-optimal stable mechanism χ_I, which is row-efficient.

We reiterate that ϕ is row-efficient because row-efficiency only requires that there exists no other allocation which every row agent *strictly* prefers with respect to the generalized max-min criterion. However, the result cannot be salvaged by strengthening row-efficiency to require that no other allocation exists which every row-agent *weakly* prefers, with at least one strictly, since, as is well-known, even the row-optimal stable mechanism violates this stronger property (see Example 2.31 of [12], for instance).

5. Conclusion

In this paper, we studied many-to-many matching under a preference restriction called the max-min criterion. Even under this restriction, we demonstrate that no stable mechanism satisfies weak Pareto efficiency, strategy-proofness, or monotonicity (respecting improvements) for agents on one side of the market. While these properties are claimed to hold for some stable mechanism by [4], this claim is incorrect, as shown by this work.

The max-min criterion is very restrictive and, as such, strong conclusions have been obtained under this restriction in past studies (see Section 2 and, in particular, footnote 3). Despite this, our results show that even the basic properties of matching mechanisms known for one-to-one settings (see [12, 6]) do not extend to many-to-many settings even under the max-min criterion.[15] In fact, for any preference criterion such that, if an agent prefers r_1 to

[15] The row-optimal stable mechanism satisfies these properties for row players in settings in which the row players have unit demand.

r_2, then that agent also prefers $\{r_1\}$ to $\{r_2\}$, there will not exist any mechanism which is column-efficient, column-monotone, or column-strategy-proof.[16] Since it seems natural that the preferences over singleton sets should reflect the underlying preferences over partners, any condition under which weak column-efficiency, column-monotonicity, and column-strategy-proofness are satisfied for some stable mechanism will be very restrictive.

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