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# Group Robust Stability in Matching Markets

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## Abstract

We propose a group robust stability notion which requires robustness against a combined manipulation, first misreporting of preferences and then rematching, by any group of students in a school choice type of matching markets. Our first result shows that there is no group robustly stable mechanism even under acyclic priority structures (Ergin (2002)). Then, we define a weak version of group robust stability, called weak group robust stability. Our main theorem shows that there is a weakly group robustly stable mechanism if and only if the priority structure is acyclic, and in that case it coincides with the student-optimal stable mechanism. Hence this result generalizes the main theorem of Kojima (2010). Then as a real-world practice, we add uncertainty regarding an acceptance of an appeal of students to rematch after the announced matching. In that setting, we show that under some conditions along with the acyclicity, the student-optimal stable mechanism is group robustly stable under uncertainty.

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# 1 Introduction

In a school choice matching problem (Abdulkadiroğlu and Sönmez (2003)), there are two finite and disjoint sets of students and schools. Each school has a capacity which limits the maximum number of students that it can be assigned to. Each student has a preference profile over schools and being unassigned, and each school has a priority order over students and keeping a seat vacant. These priority orders are determined according to certain criteria imposed by law such as proximity of students' residences to schools, which school students' siblings are attending (if applicable), etc. An allocation is a matching if no student is assigned to more than one school, and no school is assigned to more students than its capacity. A matching is stable if (i) no student prefers being unassigned to his assignment, (ii) no school prefers keeping a seat vacant to some student in its assignment, and (iii) there is no unassigned student-school pair such that the student prefers the school to his assignment and either the student has a higher priority for the school than some student in its assignment or the school has a vacant seat and it prefers being assigned to the student rather than keeping the seat vacant.

Stability has been central in matching theory. Besides its theoretical appeal, we observe how vital it is for real-world practical market designs from some widely-known markets in which previously used and failed unstable mechanisms have been replaced with stable ones which have been performing well.<sup>1</sup> For instance, Roth (1984) shows that the National Resident Matching Program (NRMP) has been using a stable mechanism instead of the previously used unstable one.

Since implementing a mechanism requires information elicitation from agents, incentive properties of the stable mechanisms have been extensively studied. Roth (1982) shows that there is no mechanism that is stable and immune to preference manipulations. However, whenever either side of a market has commonly known preferences, as in the school choice, there is a mechanism which is both strategy-proof and stable (Dubins and Freedman (1981),

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<sup>1</sup>Roth (1991) gives market examples from U.K. illustrating this point.

Roth (1982)).<sup>2</sup> Sönmez (1997) and Sönmez (1999) prove that there is no stable mechanism which is immune to capacity and pre-arranged manipulations respectively.

In some centralized matching markets, agents have a right to appeal their assignments under the announced matching to rematch. For example, in the New York City high school placements for the academic year 2003-2004, more than 5,000 students appealed their assignments, and about half of them were granted (Abdulkadiroğlu et al. (2005)). In such matching markets, one arising natural question is whether students can manipulate matching mechanisms via combined manipulations of first misreporting their preferences and then blocking (rematching) the announced matching. This kind of manipulation incentives have been overlooked in the literature until recently. Chakraborty et al. (2010) proposes a strong stability notion which takes into account the combined manipulation incentives in the matching markets with interdependent values. Kojima (2010) adopts their strong stability concept to the standard matching markets without interdependent values and proposes a robust stability notion requiring robustness against the combined manipulations.

Both Chakraborty et al. (2010) and Kojima (2010) consider the combined manipulation incentives of individual agents. However, agents in a matching market could coordinate to obtain a more favorable outcome for them. Because of these coordination possibilities, extending manipulation incentives to any group of agents rather than considering only of individuals is a common exercise in the literature. It is known that even either side of a matching market has commonly known preferences, there is no stable mechanism which is group strategy-proof. Ergin (2002) characterizes the school choice type of matching markets in which the student-optimal stable mechanism is group strategy-proof. A weak version of group strategy-proofness requiring all students in a group to be strictly better off by jointly doing a preference manipulation holds in the student-optimal mechanism in the school choice context (Dubins and Freedman (1981)). Chakraborty and Citanna (2009) extends the stability notion of Chakraborty et al. (2010) by considering the combined manipulation

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<sup>2</sup>It is the student-optimal stable mechanism in the school choice context.

incentives of any group of agents in the matching markets with interdependent values. They show the impossibility of obtaining a stable mechanism in their sense and provide some conditions under which it exists.

In the current study, we generalize the robust stability notion of Kojima (2010) in the standard school choice context without interdependent values by requiring robustness against the combined manipulations by any group of students (including individual ones). We propose a group robust stability notion taking into account the combined manipulation incentives of any group of students where each student in a group gets at least weakly better off with at least one student in the group gets strictly better off by jointly doing a combined manipulation. In contrast to the main theorem of Kojima (2010), showing the existence of a robustly stable mechanism under acyclic priority structures (Ergin (2002)), our first result shows that there is no group robustly stable mechanism even under the acyclic priority structures. Given this impossibility result even under the very demanding acyclicity condition, we define a weak version of group robust stability, called weak group robust stability, and seek a condition under which it exists rather than try to find a more stringent condition for the existence of a group robustly stable mechanism.<sup>3</sup> The only difference between the group robust stability and the weak group robust stability is that the latter (one) requires each student in a combinedly manipulating group to be strictly better off. Our main theorem shows that the acyclic priority structure (Ergin (2002)) is a necessary and sufficient condition to have a weak group robustly stable mechanism, and it coincides with the student-optimal stable mechanism. Hence, this result directly generalizes the main theorem of Kojima (2010).

The stability notions requiring robustness against the combined manipulations of Kojima (2010), Chakraborty et al. (2010), Chakraborty and Citanna (2009) and the ones mentioned above assume that any student-school pair can match with each other if it is profitable for them after the announced matching. However, presumably all appeals of students to rematch are not granted. For instance, as we point out previously, for the academic year

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<sup>3</sup>Indeed, the non-existence of a group robustly stable mechanism result continues to hold under more stringent variants of the acyclicity condition of Kesten (2006).

2003-2004 in the student placement to high schools in New York City, more than 5,000 students appealed their assignments and about half of them were granted (Abdulkadiroğlu et al. (2005)). Given that the acceptance of an appeal is not guaranteed at the ex-ante stage, we address the question of whether we could obtain a group robustly stable mechanism under uncertainty regarding the acceptance of an appeal of any student to rematch after the announced matching. In this setting, even though a student appeals his assignment, either he is rematched with some probability or not rematched but keeps his announced assignment with the remaining probability. We extend our group robust stability notion to the uncertainty case in a natural way by assuming that students are expected utility maximizers. We show that under certain bound conditions on students' cardinal utility profiles, the student-optimal stable mechanism is group robustly stable under the acyclic priority structures (Ergin (2002)) as long as the exogenously given acceptance probability of an appeal is lower than some specified bound.

At that point, one might argue that the student-optimal stable mechanism being group robustly stable under the uncertainty case is rather trivial since it is known that the student-optimal stable mechanism is group strategy-proof under the acyclic priority structures (Ergin (2002)). Hence, there has to be some student in a combinedly manipulating group being strictly worse off under the announced matching, and for that student, it would not be ex-ante profitable to do the combined manipulation with the group if the probability of the acceptance of his appeal is sufficiently low. However, if he would have been unassigned under the true preference profile then even though he gets strictly worse off under the announced matching, he presumably would be able to drop his assignment without any appeal which ultimately makes him not strictly worse off. Yet, our key lemma in obtaining the result shows that there has to be some student in a combinedly manipulating group who would have been assigned to some school under the true preference profile and gets strictly worse off under the announced matching.

## 2 Related Literature

Our paper is directly related to Kojima (2010). Given the coordination possibilities of agents in matching markets, we extend the robust stability notion of Kojima (2010) to any group of students. Indeed, manipulation incentives of the groups of agents are extensively studied in the literature. It is known that there is no group-strategy proof mechanism even either side of a market has commonly known preferences. A weak version of group strategy-proofness which requires all students in a group to be strictly better off through a joint preference manipulation holds in the student-optimal stable mechanism in the school choice context due to Dubins and Freedman (1981). This result is extended by Martinez et al. (2004), Hatfield and Kojima (2007), and Hatfield and Kojima (2008) to matching problems with contracts (Hatfield and Milgrom (2005)).

In obtaining our results, the acyclic priority structure condition (Ergin (2002)) plays a critical role. There are other related studies in the literature in which the acyclicity condition turns out to be indispensable. Kojima (2010) shows that there exists a robustly stable mechanism if and only if the priority structure is acyclic, and it coincides with the student-optimal stable mechanism; Ergin (2002) proves that the student-optimal stable mechanism is group strategy-proof if and only if the priority structure is acyclic; Kesten (2008) shows that the student-optimal stable mechanism is immune to capacity manipulation (Sönmez (1997)) if and only if the priority structure is acyclic.

Two other closely related papers are Chakraborty et al. (2010) and Chakraborty and Citanna (2009). Chakraborty et al. (2010) propose a stability concept requiring robustness against the combined manipulation incentives of individual agents in the matching markets with interdependent values, and Chakraborty and Citanna (2009) extends their notion to any group of agents in the same environment. Kojima (2010) adopts the stability notion of Chakraborty et al. (2010) to the standard matching markets with private values. The current study extends the robust stability notion of Kojima (2010) in the standard matching markets with private values. Since the modeling environment of the current study and that

of Chakraborty et al. (2010) and Chakraborty and Citanna (2009) are different, the direct comparison does not make sense. However, it is important to note that the private value assumption is critical in the main theorem of the paper (Theorem 2), since Chakraborty et al. (2010) shows that even under the acyclic priority structures, there exists no stable mechanism which is robust against the combined manipulations in the matching markets with interdependent values.

### 3 Model

A matching problem is tuple  $(S, C, P, \succ, q)$ . First two components  $S$  and  $C$  are finite and disjoint sets of students and schools respectively. Each student  $s \in S$  has a preference relation  $P_s$  which is a complete, strict, and transitive binary relation over  $C$  and being unassigned (denoted by  $\emptyset$ ). Let  $\mathcal{P}$  denote the set of all preference relations. We write  $cR_s c'$  if either  $cP_s c'$  or  $c = c'$ . The vector  $P = (P_s)_{s \in S}$  is the list of the preference profiles of the students. Each school  $c \in C$  has a priority order  $\succ_c$  which is a strict, complete and transitive binary relation over  $S$  and keeping a seat vacant (denoted by  $\emptyset$ ). The list of the priority orders of the schools is denoted by  $\succ = (\succ_c)_{c \in C}$ . The last component  $q = (q_c)_{c \in C}$  is the quota profile of the schools.

We interpret the priority orders of the schools as their preferences, and extend them to over any group of the students in the responsive (Roth (1985)) way. Formally, for a school  $c \in C$  with priority order  $\succ_c$ , for all  $J \subset S$ ;

- (i) for all  $s \in S \setminus J$ ,  $J \cup \{s\} \succ_c J$  if and only if  $\{s\} \succ_c \emptyset$
- (ii) for all  $s, s' \in S \setminus J$ ,  $J \cup \{s\} \succ_c J \cup \{s'\}$  if and only if  $\{s\} \succ_c \{s'\}$ .

The choice of a school  $c \in C$  with quota  $q_c$  from a group of students  $J \subset S$  is defined as;

$$Ch_c(\succ_c, q_c, J) = \{J' \subset J : |J'| \leq q_c, J' \succ_c J'' \text{ for all } J'' \subset J \text{ such that } |J''| \leq q_c\}.$$

A matching  $\mu$  is an allocation under which each student is assigned to at most one school, and no school is assigned to more students than its quota. We write  $\mu_i$  for the assignment of an agent  $i \in S \cup C$  under  $\mu$ . We denote the set of all matchings for a given  $q$  by  $\mathcal{M}(q)$ .

A matching  $\mu$  is individually rational if  $\mu_s R_s \emptyset$  and  $\mu_c \subset Ch_c(\succ_c, q_c, \mu_c)$  for all  $s \in S$  and  $c \in C$ . A matching  $\mu$  is blocked by a student-school pair  $(s, c) \in S \times C$  if  $c P_s \mu_s$  and  $s \in Ch_c(\succ_c, q_c, \mu_c \cup \{s\})$ . A matching  $\mu$  is stable if it is individually rational and is not blocked by any pair  $(s, c) \in S \times C$ . Since we are primarily interested in the school choice problem, for simplicity, we assume  $s \succ_c \emptyset$  for all  $s \in S$  and  $c \in C$ .

As the priority orders of the schools are exogenously given, thereby publicly known, we take the same view as Kojima (2010) and refer to a tuple  $(S, C, \succ, q)$  as a market and consider only the students' preferences privately known information. For a given  $q$ , a mechanism  $\psi$  is a function from  $\mathcal{P}^{|S|}$  to  $\mathcal{M}(q)$ . A mechanism  $\psi$  is stable if  $\psi(P)$  is stable for every  $P \in \mathcal{P}^{|S|}$ . A mechanism  $\psi$  is strategy-proof if  $\psi_s(P) P_s \psi(P'_s, P_{-s})^4$  for every  $s \in S, P \in \mathcal{P}^{|S|}$ , and  $P'_s \in \mathcal{P}$ . Mechanism  $\psi$  is group strategy-proof if there is no  $J \subset S, P \in \mathcal{P}^{|S|}$ , and  $P'_J \in \mathcal{P}^{|J|}$  such that  $\psi_s(P'_J, P_{-J}) R_s \psi_s(P)$  for all  $s \in J$ , and for some  $s' \in J, \psi_{s'}(P'_J, P_{-J}) P_{s'} \psi_{s'}(P)$ . A mechanism  $\psi$  is nonbossy if there is no  $s \in S, P \in \mathcal{P}^{|S|}$ , and  $P'_s \in \mathcal{P}$  such that  $\psi_s(P) = \psi_s(P'_s, P_{-s})$  but  $\psi(P) \neq \psi(P'_s, P_{-s})$ .

Before proceeding with the rest of the paper, we outline the deferred acceptance algorithm (Gale and Shapley (1962)) which produces a stable matching  $\psi^A(P)$  for every  $P \in \mathcal{P}^{|S|}$ .

**Step 1.** Each student applies to his first choice school. Each school that receives one or more offers holds as many best acceptable offers as at most its quota. The algorithm terminates if no offer is rejected. Otherwise it continues with the next step.

In general,

**Step  $t$ .** Each student who was rejected in step  $t - 1$  applies to his best acceptable choice in the set of schools to which he has not applied before. Each school holds as many best acceptable offers as at most its quota among the set of offers held at step  $(t - 1)$  and the

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<sup>4</sup> $P_{-s}$  denotes the preference profile of the students except student  $s$ .



offers that it receives at that step. It rejects the rest. The algorithm terminates if no offer is rejected. Otherwise, it continues with the next step.

The algorithm terminates when no student applies to a school, and the tentatively held offers at the termination step are realized as assignments.

The following robust stability notion is due to Kojima (2010).

**Definition 1.** *A mechanism  $\psi$  is robustly stable if the following conditions are satisfied.*

- (1)  $\psi$  is stable,
- (2)  $\psi$  is strategy-proof, and
- (3) there exists no  $s \in S$ ,  $c \in C$ ,  $P \in \mathcal{P}^{|S|}$ , and  $P'_s \in \mathcal{P}$  such that (i)  $cP_s\psi_s(P)$ , (ii)  $s \succ_c \emptyset$ , and (iii)  $s \succ_c s'$  for some  $s' \in \psi_c(P'_s, P_{-s})$  or  $|\psi_c(P'_s, P_{-s})| < q_c$ .

In words, a robustly stable mechanism does not allow students to manipulate the mechanism through either a preference misreporting or a combined manipulation of first misreporting then blocking (rematching) the announced matching. However, the robust stability notion ignores the manipulation incentives of any group of students but individual ones.

**Definition 2.** *A mechanism  $\psi$  is group robustly stable if the following conditions are satisfied.*

- (1)  $\psi$  is stable,
- (2) there exists no  $J \subset S$ ,  $P \in \mathcal{P}^{|S|}$ ,  $P'_J \in \mathcal{P}^{|J|}$ , a partition of  $J = \bigsqcup H_i$ <sup>5</sup>, and  $c_i \in C$  for each  $H_i$ , different across  $H_i$ , such that for all  $s \in H_i$  and for all  $H_i$ :

- (i)  $c_i R_s \psi_s(P)$  and  $H_i \subset Ch_{c_i}(\succ_{c_i}, q_{c_i}, \psi_{c_i}(P'_J, P_{-J}) \cup H_i)$
- (ii) There exists a student  $s \in H_i$  for some  $H_i$  such that  $c_i P_s \psi_s(P)$ .

**Remark 1.** “Condition 2” in the definition of the group robust stability considers both the preference manipulation and the combined manipulation incentives of any group of students. In words, a mechanism is group robustly stable if it is stable, group strategy-proof, and non-manipulable via a combined manipulation by any group of students.

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<sup>5</sup> $H_i \cap H_k = \emptyset$  for all  $i \neq k$ , and  $J = \bigcup_i H_i$

**Remark 2.** In a combinedly manipulating group, all students in the group do not necessarily engage in rematching after the announced matching, as there might be some student who is already at least weakly better off under the announced matching. However, by the definition of the combined manipulation, there has to be some student in a combinedly manipulating group rematching after the announced matching since otherwise it would collapse to a pure preference manipulation.

Kojima (2010) shows that there is no robustly stable mechanism, and since group robust stability notion is a generalization of robust stability, the non-existence result also holds for the group robust stability.

**Fact 1.** *There exists no mechanism that is group robustly stable.*

The following definition of an acyclic priority structure, which will play a key role in the rest of the paper, is due to Ergin (2002).

**Definition 3.** (Ergin (2002)). *Let  $(\succ, q)$  be a priority structure. A cycle is  $a, b \in C$ ,  $i, j, k \in S$  such that;*

- (i)  $i \succ_a j \succ_a k \succ_a \emptyset$  and  $k \succ_b i \succ_b \emptyset$ ,
- (ii) *There exist disjoint sets of students  $S_a, S_b \subset S \setminus \{i, j, k\}$  such that  $|S_a| = q_a - 1$ ,  $|S_b| = q_b - 1$ ,  $s \succ_a j$  for every  $s \in S_a$ , and  $s \succ_b i$  for every  $s \in S_b$ .*

A priority structure  $(\succ, q)$  is acyclic if there exists no cycle.

The main theorem of Kojima (2010) shows that acyclicity is a necessary and sufficient condition for the existence of a robustly stable mechanism. However, the following theorem shows that this is not the case for group robust stability.

**Theorem 1.** *There exists no mechanism that is group robustly stable even under the acyclic priority structures.*

*Proof.* Let  $\psi$  be a stable mechanism (we only consider the stable mechanisms since we require stability from a group robustly stable mechanism).

Consider a problem with  $S = \{i, j, k\}$ ,  $C = \{a, b\}$ ,  $q_a = q_b = 1$ , and

$i \succ_a k \succ_a j \succ_a \emptyset$ ;  $k \succ_b i \succ_b j \succ_b \emptyset$ .

$P_i : a, \emptyset$ ;  $P_j : b, \emptyset$ ;  $P_k : a, b, \emptyset$ .

Note that the above priority structure  $(\succ, q)$  is acyclic. Under the true preference profile  $P = (P_i, P_j, P_k)$ ,  $\psi(P) = (\psi_i(P), \psi_j(P), \psi_k(P)) = (a, \emptyset, b)$ . Students  $i$  and  $j$  can form a group and combinedly manipulate the matching outcome if student  $i$  misreports his preference by reporting  $P'_i : \emptyset$ . Under the preference profile  $P' = (P'_i, P_j, P_k)$ ,  $\psi(P') = (\emptyset, b, a)$ . Since  $i \succ_a k$ , student  $i$  blocks the matching  $\psi(P'_i, P_j, P_k)$ , and thereby he ends up with “school  $a$ ” through the rematching after the announced matching  $\psi(P'_i, P_j, P_k)$ , while student  $j$  gets strictly better off under  $\psi(P'_i, P_j, P_k)$ . ■

Given the above non-existence result even under the very demanding acyclicity condition, we weaken the group robust stability notion and seek a condition under which it exists rather than try to find a more stringent condition than acyclicity under which a group robustly stable mechanism exists.<sup>6</sup>

**Definition 4.** *A mechanism  $\psi$  is weakly group robustly stable if the following conditions are satisfied.*

- (1)  $\psi$  is stable,
- (2)  $\psi$  is group strategy-proof,
- (3) *There exists no  $J \subset S$ ,  $P \in P^{|S|}$ ,  $P'_J \in P^{|J|}$ , a partition of  $J = \bigsqcup H_i$ , and  $c_i \in C$  for each  $H_i$ , different across  $H_i$ , such that for all  $s \in H_i$  and for all  $H_i$ :*

$$(i) c_i P_s \psi_s(P) \quad \text{and} \quad (ii) H_i \subset Ch_{c_i}(\succ_{c_i}, q_{c_i}, \psi_{c_i}(P'_J, P_{-J}) \cup H_i)$$

**Remark 3.** In the definition of the weak group robust stability, we write group strategy-proofness requirement separately, since “condition 3” excludes the preference manipulation

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<sup>6</sup>Indeed, the example given in the proof of the theorem 2 shows that the non-existence of a group robustly stable mechanism result continues to hold even under more stringent variants of the acyclicity condition of Kesten (2006).

incentives of any group of students where all students in a manipulating group get weakly better off, and some student(s) in the group (but not all of them) gets strictly better off.

The only difference between the group robust stability and the weak group robust stability is that the latter (one) requires each student in a combinedly manipulating group to be strictly better off <sup>7</sup>, whereas the former (one) requires at least one student in a combinedly manipulating group to be strictly better off, while all students in the group get at least weakly better off.

It is immediate to conclude that there is no weakly group robustly stable mechanism due to the non-existence of a robustly stable mechanism (Kojima (2010)).

**Fact 2.** *There is no weakly group robustly stable mechanism.*

The following main result of the paper characterizes the market conditions under which a weakly group robustly mechanism exists.

**Theorem 2.** *Given a market  $(S, C, \succ, q)$ , the followings are equivalent.*

- (1) *There exists a weakly group robustly stable mechanism.*
- (2)  *$\psi^A$  is weakly group robustly stable.*
- (3) *The priority structure  $(\succ, q)$  is acyclic.*

*Proof.* See Appendix. ■

In the next section, we address the question of whether we could obtain a group robustly stable mechanism if we add uncertainty to the matching systems as a real-world practice.

## 4 Adding Uncertainty

Presumably, all appeals of students to their assignments after the announced matching are not accepted. For example, as pointed out before, for the academic year 2003-2004 in

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<sup>7</sup>This way of weakening group strategy-proofness is common in the literature. In the school choice context, it is known that the student-optimal stable mechanism is not group strategy proof. However, if we relax the group strategy-proofness in this way, then the student-optimal stable mechanism is group strategy-proof in this weak sense (Dubins and Freedman (1981)).

student placements to New York City high schools, more than 5,000 students appealed their assignments, and about half of them were granted. Hence, from the ex-ante point of view, we can think of a situation where students are not sure whether they could manage to rematch after the produced matching. In order to model this, we add uncertainty to the matching systems regarding the acceptance of an appeal to rematch after the announced matching. In our model under uncertainty, there is a commonly known exogenous probability of the chance of a student to rematch upon an appeal from him.<sup>8</sup> Therefore, each appealing student is either rematched with the exogenously given probability or keeps his announced assignment with the remaining probability. We extend the group robust stability notion to the uncertainty case in a natural way by assuming students being expected utility maximizers. The main theorem of this section shows that under some conditions along with acyclicity, the student-optimal stable mechanism is “group robustly stable under uncertainty”. In what follows, we formalize the school choice under uncertainty and demonstrate our results.

A school choice matching problem under uncertainty is tuple  $(S, C, P, u, \succ, q, \lambda)$ . The fourth component  $u = (u_s)_{s \in S}$  is the cardinal utility preference profile of students  $S$  where each  $u_s$  is a function from the set  $C \cup \emptyset$  to  $\mathbb{R}$  representing the ordinal preference profile  $P_s$ .<sup>9</sup> The last component  $\lambda \in (0, 1)$  is the exogenous probability of the acceptance of an appeal to rematch after a given matching. All the other components are the same with those without uncertainty.

We refer to  $(S, C, \succ, q, \lambda)$  as a market. As in the deterministic case, only the preferences (both cardinal and ordinal) of students are private information. As a real-world practice, students report their ordinal preference profile rather than their cardinal utility preference profile to the matching mechanisms.

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<sup>8</sup>This kind of probabilistic acceptance scheme could arise in a case where the social planner decides whether or not to accept an appeal according to a realization of some randomization device. Note that the realizations have to be independent.

<sup>9</sup> $u_s$  represents  $P_s$ , if  $u_s(c) > u_s(c') \Leftrightarrow c P_s c'$  for all  $c, c' \in C \cup \emptyset$ .

**Definition 5.** A mechanism is group robustly stable under uncertainty if the following conditions are satisfied.

(i)  $\psi$  is stable,

(ii) there exists no  $J \subset S$ ,  $P \in \mathcal{P}^{|S|}$ ,  $P'_J \in \mathcal{P}^{|J|}$ , a partition of  $J = \bigsqcup H_i$ , and  $c_i \in C$  for each  $H_i$ , different across  $H_i$ , such that for all  $s \in H_i$  and for all  $H_i$ :

**Condition A** (i)  $\lambda u_s(c_i) + (1 - \lambda)u_s(\psi_s(P'_J, P_{-J})) \geq u_s(\psi_s(P))$ , and

(ii)  $H_i \subset Ch_{c_i}(\succ_{c_i}, q_{c_i}, \psi_{c_i}(P'_J, P_{-J}) \cup H_i)$

**Condition B** There exists a student  $s \in H_i$  for some  $H_i$  such that;

$\lambda u_s(c_i) + (1 - \lambda)u_s(\psi_s(P'_J, P_{-J})) > u_s(\psi_s(P))$ .

**Remark 5.** Note that the case of “ $\lambda = 1$ ” collapses to the deterministic case which we consider previously in the paper. As in the deterministic case, the above definition requires non-manipulability by any group of students via a preference manipulation as well as a combined manipulation. The group strategy-proofness requirement corresponds the case where the students’ target schools through appealing are the same with their assignment under the announced matching.

**Remark 6.** The only students in a combinedly manipulating group for whom uncertainty matters are the ones who are strictly worse off under the announced matching. Through appealing, they ultimately either keep their announced assignment with some pre-specified probability or rematch and get at least weakly better off with the remaining probability. Thus, the combined manipulation might not be ex-ante profitable for these students.

In what follows, we prove that under some conditions on the cardinal utility profiles of students, the student-optimal stable matching is group robustly stable under the acyclic priority structures as long as the given probability of the acceptance of an appeal is lower than some specified bound. As we mention in the introduction section of the paper, it might be seen rather trivial result, since we know that the student-optimal mechanism is group strategy-proof under acyclicity which guarantees the existence of a student in a combinedly

manipulating group who get strictly worse off under the announced matching. Therefore, under a sufficiently low acceptance probability, the combined manipulation would not be ex-ante profitable for him. However, students presumably can drop their assignments and be unassigned without any appealing procedure. Therefore, if he would have been unassigned under the true preference profile then basically by dropping his assignment under the announced matching he can make himself to be not worse off through the combined manipulation. Yet, our key lemma in the proof of the theorem 3 shows that there has to be some student in a combinedly manipulation group who would have been assigned to some school under the true preference profile and gets strictly worse off under the announced matching.

**Lemma 1.** *Under the acyclic priority structures, if the “condition 2” in the group robust stability definition (without uncertainty) holds for a group of students  $J \subset S$  with  $P \in \mathcal{P}^{|S|}$ , and  $P'_J \in \mathcal{P}^J$ , then there exists a student  $s \in J$  such that  $\emptyset \neq \psi_s^A(P)P_s\psi_s^A(P'_J, P_{-J})$ .*

*Proof.* See Appendix. ■

**Theorem 3.** *Under the normalization  $u_s(\emptyset) = 0$  for all  $s \in S$ , if  $|u_s(c)| < k$ , and  $|u(c) - u(c')| > h$ , for all  $s \in S$ , for all  $c, c' \in C$ , for some  $k, h \in \mathbb{R}_{++}$ , then under the acyclic priority structures  $\psi^A$  is group robustly stable under uncertainty for all  $\lambda < \frac{h}{k}$ .*

*Proof.* See Appendix. ■

## 5 Conclusion

Motivated by the recent line of research due to Chakraborty et al. (2010), Chakraborty and Citanna (2009), and Kojima (2010) on the combined manipulations, this paper complements their studies by analyzing the combined manipulation incentives of any group of students in the standard matching markets without interdependent values. We propose the (weak) group robust stability notions which are extended versions of the robust stability notion of Kojima (2010) to any group of students. We show that under the acyclic priority

structures, in contrast to the existence of a robustly stable mechanism (Kojima (2010)), there is no group robustly stable mechanism. Then, we weaken the group robust stability notion in a common way as in the literature. Our main result shows that the acyclic priority structure is a necessary and sufficient condition for obtaining a weak group robustly stable mechanism, and in that case it coincides with the student-optimal stable mechanism. Thereby, our main theorem generalizes the main theorem of Kojima (2010). Then as a real-world practice, we add uncertainty into the model. After having extended the group robust stability notion to the uncertainty case in a natural way, we show that under some conditions along with the acyclicity, the student-optimal stable mechanism is group robustly stable under uncertainty.

Our analysis suggests that it is not possible to avoid the combined manipulations if students could cooperate with each other even under the very demanding acyclic priority structure condition. However, if the social planner announces a sufficiently low probability of the acceptance of an appeal before the preference reporting stage, group robust stability is obtained under certain domains as long as acyclicity holds. On the other hand, if students are assumed to do a combined manipulation only if they would be strictly better off then the social planner needs to influence the priority structure of schools in a way that makes it acyclic in order to avoid the combined manipulations.

## Appendix

*Proof of Theorem 2.* Due to the theorem 2 of Kojima (2010), it suffices to prove only (3) implies (2).

(3)  $\Rightarrow$  (2). Due to Ergin (2002),  $\psi^A$  is group strategy-proof under the acyclic priority structures. Moreover, since  $\psi^A$  is stable it is enough to show that  $\psi^A$  is immune to the combined manipulation incentives of any group of students. Suppose  $\psi^A$  is not weakly group robustly stable. Then this implies that the following condition holds;

**Condition C.** There exists  $J \subset S$ ,  $P \in P^{|S|}$ ,  $P'_j \in P^{|J|}$ , a partition of  $J = \bigsqcup H_i$  and  $c_i \in C$  for each  $H_i$ , different across  $H_i$ , such that for all  $s \in H_i$  and for all  $H_i$ ,



(i)  $c_i P_s \psi_s(P)$  and (ii)  $H_i \subset Ch_{c_i}(\succ_{c_i}, q_{c_i}, \psi_{c_i}(P'_J, P_{-J}) \cup H_i)$ .

Let  $P' = (P'_J, P_{-J})$ . Pick the student  $s_0 \in J$  who is the last tentatively matched student among the set of students  $J$  in  $\psi^A$  under the preference profile  $P$  (if there are more than one such student then pick anyone among them). Let  $P_s^{\psi^A}$  denote the preference profile of a student  $s \in J$  according to which the relative ranking of all strictly preferred schools to  $\psi_s^A(P)$  with respect to  $P_s$  are the same with  $P_s$ , and all other schools are unacceptable.

Consider the problem instance where the preference profile is  $P_0 = (P_{J \setminus \{s_0\}}, P_{s_0}^{\psi^A}, P_{-J})$ . Note that by the choice of student  $s_0$ , we have  $\psi_s^A(P_0) = \psi_s^A(P)$  for all  $s \in J \setminus \{s_0\}$  and  $\psi_{s_0}^A(P_0) = \emptyset$ .

Let  $\hat{P}_s$  denote the preference profile of a student  $s$  under which every school is unacceptable, and consider the preference profile  $P_1 = (P_{J \setminus \{s_0\}}, \hat{P}_{s_0}, P_{-J})$ . Given that  $\psi^A$  is nonbossy under the acyclic priority structures (Ergin (2002)),  $\psi^A(P_0) = \psi^A(P_1)$  as  $\psi_{s_0}^A(P_0) = \psi_{s_0}^A(P_1)$ .

We claim that at the matching problem instance where the students' preference profile is  $P_1$ , the group of students  $J \setminus \{s_0\}$  has false preference profile  $\hat{P}_{J \setminus \{s_0\}} = (\hat{P}_s)_{s \in J \setminus \{s_0\}}$  satisfying the condition C. Let  $\hat{P}_J = (\hat{P}_s)_{s \in J}$ . In order to prove the claim, firstly observe that due to the widely-known comparative statistics result (Gale and Sotomayor (1985))<sup>10</sup>, we have  $\psi_s^A(\hat{P}_J, P_{-J}) R_s \psi_s^A(P'_J, P_{-J})$  for all  $s \in S \setminus J$ . Furthermore, for all school  $c \in C$ , there is no  $s \in \psi_c^A(\hat{P}_J, P_{-J}) \setminus \psi_c^A(P'_J, P_{-J})$ <sup>11</sup> such that  $s \succ_c s'$  for some  $s' \in \psi_c^A(P'_J, P_{-J})$ . This is true since  $\psi^A$  is stable and  $\psi_s^A(\hat{P}_J, P_{-J}) R_s \psi_s^A(P'_J, P_{-J})$  for all  $s \in S \setminus J$ . On the other hand, since  $\psi_s^A(P_1) = \psi_s^A(P)$  for all  $s \in J \setminus \{s_0\}$  we conclude that our claim is true, that is at the matching problem instance where the preference profile is  $P_1$ , the group of students  $J \setminus \{s_0\}$  has false preference profile  $\hat{P}_{J \setminus \{s_0\}}$  satisfying condition C.

Redoing the same procedure, but now starting with the problem where the students' preference profile is  $P_1$ , the group of students is  $J \setminus \{s_0\}$ , and the false preference profile for

<sup>10</sup>In words, for our case, as the set of students in a matching market gets smaller in the subset relation sense no remaining student in the matching market will be strictly worse off and no school will be strictly better off under the student-optimal stable mechanism.

<sup>11</sup> $\psi_c^A$  stands for the assignment of school  $c$ .

that group satisfying condition C is  $\hat{P}_{J \setminus \{s_0\}}$  gives us a problem instance where the group of students  $J \setminus \{s_0, s_1\}$ ,  $s_1$  is the student who is the last tentatively matched student in  $\psi^A$  among the group  $J \setminus \{s_0\}$  under  $P_1$ , has a false preference profile satisfying condition C. Continuing in the same manner gives us a problem instance where there exists an individual student  $s$  and a false preference profile  $P'_s$  satisfying condition C. However, this contradicts the existence of a robustly stable mechanism under the acyclic priority structures result of Kojima (2010). ■

*Proof of Lemma 1.* Assume that there exists  $J \subset S, P \in \mathcal{P}^{|S|}$ , and  $P'_J \in \mathcal{P}^{|J|}$  satisfying condition 2 in the group robust stability definition (without uncertainty). Let  $P' = (P'_J, P_{-J})$ .

If  $\psi_s^A(P') R_s \psi_s^A(P)$  for all  $s \in J$  with at least one student for whom it holds strictly, then this contradicts the group strategy-proofness of  $\psi^A$  since under the acyclic priority structures  $\psi^A$  is group strategy-proof (Ergin (2002)). (Recall that as we point out in the Remark 1, condition 2 takes into account the preference manipulations by any group of students as well as the combined manipulations.)

If  $\psi_s^A(P') = \psi_s^A(P)$  for all  $s \in J$ , then we claim that  $\psi^A(P') = \psi^A(P)$ . In order to prove the claim, let define  $A = \{s \in S \setminus J : \psi_s^A(P') P_s \psi_s^A(P)\}$  and suppose that  $\psi^A(P') \neq \psi^A(P)$ . If  $A = \emptyset$ , then this implies that  $\exists \hat{s} \in S \setminus J$  such that  $\psi_{\hat{s}}^A(P) P_{\hat{s}} \psi_{\hat{s}}^A(P')$  (since we assume  $\psi^A(P') \neq \psi^A(P)$ ). Moreover, for all  $s \in S \setminus \{J \cup \{\hat{s}\}\}$ ,  $\psi_s^A(P) R_s \psi_s^A(P')$ . On the other hand by the assumption, we have  $\psi_s^A(P') = \psi_s^A(P)$  for all  $s \in J$ . Hence, at the matching problem instance where the preference profile is  $P' = (P'_J, P_{-J})$ , we have  $\psi_s^A(P) R'_s \psi_s^A(P')$  for all  $s \in J$  and  $\psi_s^A(P) R_s \psi_s^A(P')$  for all  $s \in S \setminus J$  and with at least one student  $\hat{s}$ , it holds strictly. However, this contradicts the fact that  $\psi^A$  is efficient under the acyclic priority structures (Ergin (2002)). For the case of  $A \neq \emptyset$ , let  $\hat{s} \in A$ . Then, we have  $\psi_{\hat{s}}^A(P') P_{\hat{s}} \psi_{\hat{s}}^A(P)$  and, by assumption,  $\psi_s^A(P') = \psi_s^A(P)$  for all  $s \in J$ . However, this contradicts the group-strategy proofness of  $\psi^A$  under the acyclic priority structures result (Ergin (2002)) since the group of students  $J \cup \{\hat{s}\}$  can jointly manipulate the mechanism  $\psi^A$  by misreporting their preferences. Therefore, we conclude that  $\psi^A(P') = \psi^A(P)$ , but in that case this contradicts

the assumption that  $P'$  satisfies the condition 2.

The above analyses prove that there exists  $s \in J$  such that  $\psi_s^A(P)P_s\psi_s^A(P')$ . Let define

$$W = \{s \in J : \psi_s^A(P)P_s\psi_s^A(P')\}.$$

In order to finish the proof, we need to show that there exists some  $s \in W$  such that  $\psi_s^A(P) \neq \emptyset$ .

Suppose that for all  $s \in W$ ,  $\psi_s^A(P) = \emptyset$ . Let  $P''_J$  be a preference profile defined as follows;

$$\begin{aligned} P''_s &: \psi_s^A(P'), \emptyset, \text{ for all } s \in J \setminus W, \\ P''_s &: \emptyset, \text{ for all } s \in W. \end{aligned}$$

Let  $P'' = (P''_J, P_{-J})$ . By the widely-known comparative statistics result (Gale and Sotomayor (1985))<sup>12</sup>, we have  $\psi_s^A(P'')R_s\psi_s^A(P')$  for all  $s \in J \setminus W$  and  $\psi_s^A(P'')R_s\psi_s^A(P')$  for all  $s \in S \setminus J$ . This implies that for all  $c \in C$ , there is no  $s \in \psi_c^A(P'') \setminus \psi_c^A(P')$  such that  $s \succ_c s'$  for some  $s' \in \psi_c^A(P')$ . Since, otherwise,  $\psi^A$  could not have been stable at the problem where the students' preference profile is  $P'$  (note that if there exists such a student  $s$  then  $s \in S \setminus W$ ). Hence, preference profile  $P''_J$  for the group of students  $J$  satisfies the condition 2 in the given original problem where the students' preference profile is  $P$ .

Note that by the definition of  $P''$ , we have  $\psi_s^A(P'') = \psi_s^A(P')$  for all  $s \in J \setminus W$  and  $\psi_s^A(P'') = \emptyset$  for all  $s \in W$ . Hence,  $\psi_s^A(P'')R_s\psi_s^A(P)$  for all  $s \in J$  and  $P''$  satisfies the condition 2. However, we show the impossibility of this case in the first two analyses in the proof. ■

*Proof of Theorem 3.* Assume that the assertion of the theorem is not true, this implies that under the given conditions,  $\psi^A$  is not group robustly stable mechanism under uncertainty for some  $\lambda < \frac{h}{k}$ . Since  $\psi^A$  is stable, we conclude that there exists  $J \subset S$ ,  $P \in \mathcal{P}^{|S|}$ ,  $P'_J \in \mathcal{P}^{|J|}$ , a partition of  $J = \bigsqcup H_i$ , and  $c_i \in C$  for each  $H_i$ , differs across  $H_i$ , such that condition A, and condition B are met.

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<sup>12</sup>Refer to footnote 10 for the statement.

By Lemma 1, there exists  $s \in J$  such that  $\emptyset \neq \psi_s^A(P)P_s\psi_s^A(P'_J, P_{-J})$ . Since  $P'_J$  satisfies condition A by our assumption we have

$$\lambda u_s(c_s) + (1 - \lambda)u_s(\psi_s^A(P'_J, P_{-J})) \geq u_s(\psi_s^A(P)) \quad (1)$$

where  $c_s$  is the school for which student  $s$  aims through the combined manipulation.

However, by the conditions on cardinal utility profiles in the statement of the theorem,

$$\lambda u_s(c_s) < h \text{ and } u_s(\psi_s^A(P)) - (1 - \lambda)u_s(\psi_s^A(P'_J, P_J)) > h$$

which contradicts the inequality 1. ■

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