The Role of Annuitized Wealth in Post-Retirement Behavior*

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September 30, 2016

Abstract

This paper develops a tractable model of post-retirement behavior with health status uncertainty and state verification difficulties. The model distinguishes between annuitized and non-annuitized wealth and features Medicaid assistance with nursing-home care. We show how to solve the potentially complex dynamic problem analytically, making it possible to characterize optimal behavior with phase diagrams. The analysis provides an integrated treatment of portfolio composition and consumption/wealth accumulation choices. The model can explain differences in post-retirement saving behavior conditional on initial wealth level, as well as account for retirees’ reluctance to fully annuitize their liquid wealth.

1 Introduction

Interest in the life-cycle behavior of retired households has increased with population aging and the associated strain on public programs for the elderly. Yet post-retirement behavior has proved challenging to understand. Standard theories, for example, are hard to reconcile with evidence that shows a lack of wealth depletion after retirement — the “retirement-saving puzzle” — and a low demand for annuities at retirement — the “annuity puzzle.” Analytic difficulties emerge as well. Some come from the fact that social insurance programs for older people tend to have elaborate rules, and the incentives that these rules generate often cannot be studied with the standard toolkit. Other difficulties arise from interactions of health uncertainty with incomplete financial and insurance markets. The purpose of this paper is to develop a parsimonious model that incorporates important features of the economic environment, yet retains sufficient tractability to

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*The authors thank Andrew Caplin and Matthew Shapiro, as well as seminar participants at University of California Santa Barbara, MRRC Research Workshop, NBER Summer Institute, BYU Computational Economics Conference, Kansai University Osaka, NETSPAR Conference Amsterdam, and CIREQ Workshop Montreal. This work was supported by NIH/NIA grant R01-AG030841-01. The opinions and conclusions are solely those of the authors and should not be considered as representing the options or policy of any agency of the Federal Government.

1E.g., Hubbard et al. [1994, 1995], Palumbo [1999], Sinclair and Smetters [2004], Reichling and Smetters [2013], Dynan et al. [2004], Scholz et al. [2006], Scholz and Seshadri [2009], Ameriks et al. [2011, 2015a, 2015b], DeNardi et al. [2010, 2013], Lockwood [2014], Love et al. [2009], Laibson [2011], Finkelstein et al. [2011], and Poterba et al. [2011, 2012], Pashchenko [2013].
be useful for qualitative, as well as quantitative, analysis. The model emphasizes the distinction between annuitized and non-annuitized wealth. With it, we are able to study the mechanisms through which portfolio composition interacts with public programs and affects retiree behavior.

The model captures uncertain health and the correlation of major health changes with changes in mortality risk. Importantly, it assumes informational asymmetries that lead to incomplete private markets for long-term care insurance. It also incorporates a means-tested public alternative, Medicaid nursing-home care, which households can use as a fall-back during poor health. The model takes into account the inflexible nature of annuities as a form of wealth, as well as their treatment under Medicaid.

Despite its richness, the model is analytically tractable. One key to the tractability is the model’s continuous-time formulation, which enables it to sidestep technical challenges related to non-convexities (challenges arising from the Medicaid means test — and leading most of the literature to numerical analysis). A second key is the case-by-case analytic approach that our formulation allows: although the model’s elements and assumptions generate a variety of optimal behavioral patterns, we can partition the domain of observable initial conditions in such a way that outcomes are relatively straightforward on each (partition) element.

We use the model to study two related topics. The “retirement-saving puzzle,” to take the first example, has bedeviled analysts of the basic life-cycle model of household behavior for decades (see the literature review below). The puzzle is that, in practice, a cohort’s average (non-annuitized) wealth often remains roughly constant, or even rises, long into retirement. This seemingly contradicts a core idea of the life-cycle model, namely, that households save during working years in order to dissave thereafter.

Section 5 shows that our formulation suggests a possible resolution of the inconsistency between the standard model and empirical cohort wealth trajectories. Our households begin retirement in good health but subsequently pass into lower health status, and then death. On the one hand, if needs for extra personal services raise the marginal utility of expenditure during poor health, we show that high-health-status retirees may husband wealth for the future, or even continue saving. Purchasing long-term care insurance would be a preferred alternative, but in our framework, asymmetric information (about health status) precludes complete markets. On the other hand, although a cohort’s members all eventually drop to poor health, the outflow from poor health to mortality can actually sustain the fraction of survivors in good health at a relatively high level. We show that the combination of the evolution of average health status and incentives to self-insure can dramatically influence cohort trajectories of average wealth. Options for Medicaid nursing-home care introduce further complications, but our partition of cases (see above) enables us to cope. Although essential insights of the standard life-cycle model remain, elements that this paper adds to the framework — namely, changing health status, incomplete security markets, and options for public assistance — affect household behavior in ways that can yield surprising outcomes.

Second, economists have long been interested in explanations for households’ apparent reluctance to annuitize all, or most, of their wealth at retirement. Households, for instance, often claim Social Security benefits at or below the age for full retirement benefits, thereby forgoing additional actuarially fair annuitization (Brown [2007]). To study this issue, Section 6 departs from our benchmark specification, in which annuities are given by initial endowments, and instead allows households the chance to adjust their portfolio composition optimally at retirement.

Again, we show that our formulation can shed light on otherwise confusing practical evidence. While households with low lifetime resources find end-of-life Medicaid care acceptable, the middle
class is ambivalent. Middle-class households attempt to use their private wealth to delay the standard of living that Medicaid entails — though they reserve, given uncertain longevity, Medicaid as a fall-back option. As above, asymmetric information precludes the existence of sophisticated contingent securities. Middle class households in good health then choose portfolios with a mixture of simple (i.e., non-health-contingent) annuities and bonds. (They liquidate the bonds after the arrival of poor health, turning to Medicaid after the bonds are exhausted.) In this way, a substantial demand for liquid wealth can arise among the healthy. Less than complete annuitization at retirement, at least among the middle class, may not be as puzzling as the standard life-cycle model would imply.

Section 7 returns to our baseline modeling specification. It examines the implications of our analysis for the timing of household take-up of Medicaid nursing-home care in practice. The model shows, for example, a dichotomy: low-resource household tend to accept Medicaid promptly after their health status declines, but middle-class households accumulate the means to delay Medicaid (reducing the odds that they will survive to draw upon it). Section 7 also briefly considers the bequests that emerge from self-insurance behavior.

Most of this paper’s analysis is qualitative, derived from the phase diagrams of our model. Sections 5-6, however, provide several numerical examples that illustrate potential quantitative magnitudes as well.

In the end, our analysis offers a unified explanation for two long-standing puzzles arising from the standard life-cycle model. We modify the standard life-cycle model to include multiple health states, with correspondingly varying expenditure needs; asymmetric information, leading to incomplete financial markets; and, an option for long-term care from Medicaid, subject to a means test. In all cases, we find that the new elements jointly affect household optimal portfolio choices (over annuities and liquid assets) and dynamic allocations for consumption expenditure and wealth accumulation.

1.1 Relation to the literature

This subsection describes the two puzzles above in slightly more detail and compares our approach to other recent work.

In the standard life-cycle model, households smooth their lifetime consumption by accumulating wealth prior to retirement and decumulating it thereafter. At least since Mirer [1979], evidence has seemed at variance with the model’s post-retirement prediction. Kotlikoff and Summers noted, “Decumulation of wealth after retirement is an essential aspect of the life cycle theory.

Yet simple tabulations of wealth holdings by age ... or savings rates by age ... do not support the central prediction that the aged dissave. [1988, p.54]

Recent work with panel data confirms that mean and median cohort wealth, for either singles or couples, can be stationary or rising for many years after retirement (Poterba et al. [2010]).

Recent analyses of post-retirement saving such as Ameriks et al. [2011, 2015a, 2015b] and DeNardi et al. [2010, 2013, 2015] include a number of the same elements as our framework, namely,

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2 See also, for instance, Ameriks et al. [2015], who observe, “The elementary life-cycle model predicts a strong pattern of dissaving in retirement. Yet this strong dissaving is not observed empirically. Establishing what is wrong with the simple model is vital ....” See also DeNardi et al. [2013, fig.7] as well as Smith et al. [2009], Love et al. [2009], and many others.
health changes and mortality risk, out-of-pocket expenses in poor health, government guaranteed consumption floors (in our case, Medicaid nursing-home care), and fixed annuity income. Since consumption floors can induce non-convexities, Ameriks et al. and DeNardi et al. rely upon numerical solutions. In explaining household wealth trajectories, both recognize the potential importance of post-retirement precautionary saving.

As stated, our formulation sidesteps non-convexities. The advantage is that the solution can be characterized with first-order conditions that can provide intuitions and comparative-static results. Non-convexities also raise the possibility that households will seek actuarially fair gambles to maximize their lifetime utility (e.g., Laitner [1988]), and our approach avoids that complication.

What is more, our model offers several important refinements for the study of precautionary saving. On the one hand, we show that a (healthy) household’s desire to save after retirement depends upon its portfolio composition: given two healthy households with identical total net worth, our model shows that the one with the higher fraction of annuities in its portfolio is the more likely to continue saving. On the other hand, the analysis explains why the behavior of households in good health is pivotal in driving cohort average wealth upward long into retirement.

DeNardi et al. [2013] present evidence that wealthier households tend to access Medicaid assistance later in life. Our results are consistent with this finding, and we can characterize Medicaid take-up timing analytically and provide further interpretations of the data.

Ameriks et al., DeNardi et al., and Lockwood [2014] consider the possible role of intentional bequests in sustaining private wealth holdings late in life — i.e., in helping to explain the “retirement-saving puzzle.” Our analysis, in contrast, does not require intentional bequests to fit the same evidence. Other than for the wealthiest decile of households (see Section 6), bequests that emerge in our model are by-products of incomplete annuitization.

One interpretation is that our work shows that intentional bequests are not needed for analyzing this paper’s issues and so, in the spirit of Occam’s razor, we omit them. Another is as follows. Survey evidence on intentional bequests is mixed: respondents to direct questions about leaving a bequest split approximately equally between answering that bequests are important and not important (Lockwood [2014], Laitner and Juster [1996]). Our analysis allows one to rationalize the post-retirement behavior of the latter group (as well as those for whom an “important” bequest could be a modest family heirloom).

Since the seminal work of Yaari [1965], many economists have sought explanations for why households do not fully annuitize their private wealth at retirement. Benartzi et al. write,

“The theoretical prediction that many people will want to annuitize a substantial portion of their wealth stands in sharp contrast to what we observe. [2011, p.149]

There is a rich literature on this “annuity puzzle” (e.g., Finkelstein and Poterba [2004], Davidoff et al. [2005], Mitchell et al. [1999], Friedman and Warshawski [1990], Benartzi et al. [2011], and many others).

Both this paper and Reichling and Smetters [2015] offer new interpretations of the “annuity puzzle.” While the studies have a number of assumptions in common, the institutional settings differ. Reichling and Smetters allow a household whose current health and/or mortality hazards have changed to purchase new annuities reflecting the revised status. In our model, state-verification problems preclude health-contingent annuities. Nonetheless, a household suffering a decline in health status can access Medicaid nursing-home care, and that option alone, we show, can substantially reduce the demand for annuities at retirement.
Some explanations of the “annuity puzzle” (e.g., Friedman and Warshawsky [1990]) give intentional bequests a prominent role. As in the case of the retirement-saving puzzle, our analysis does not rely upon intentional bequests.

Ameriks et al. [2015a] present simulations of a formulation that has health changes and state-dependent utility. Given a 10% load factor on annuities and households with $50-100,000 of existing income and bond wealth up to $400,000, they find essentially no demand for extra annuities at retirement (Ameriks et al. [2015a, fig.10]). We show that this outcome is consistent with the qualitative implications of our model, and we show how and why household initial conditions, health-status realizations, and interest rates affect outcomes.

The organization of this paper is as follows. Section 2 presents our assumptions and compares our formulation with others in the literature. Sections 3-4 analyze our model. Section 5 considers the retirement saving puzzle, Section 6 the annuity puzzle, and Section 7 Medicaid take-up and bequests. Section 8 concludes.

2 Model

As indicated in the introduction, we follow the recent literature in subdividing a household’s post-retirement years into intervals with good and poor health.

We study single-person, retired households. At any age $s$, a household’s health state, $h$, is either “high,” $H$, or “low,” $L$. The household starts retirement with $h = H$. There is a Poisson process with hazard rate $\lambda > 0$ such that at the first Poisson event the health state drops to low. Once in state $h = L$, a second Poisson process begins, with parameter $\Lambda > 0$. At the Poisson event for the second process, household’s life ends.

We focus on the general “health state” of an individual, rather than his/her medical status. Think of “health state” as referring to chronic conditions. Consider, for example, troubles with activities of daily living (ADLs), such as eating, bathing, dressing, or transferring in and out of bed. Individuals with such difficulties may need to hire assistance or move to a nursing home. The expense can be substantial. It may, in practice, be the largest part of average out-of-pocket (OOP) medical expenses (see, for instance, Marshall et al. [2010], Hurd and Rohwedder [2009]).

State-dependent utility We assume that health state affects behavior through state-dependent utility. In our framework, there are no direct budgetary consequences from changes in $h$ – all retirees have access to Medicare insurance that covers the medical part of long-term care needs. By contrast, we treat all non-medical long-term care (LTC) expenses (i.e., health-related expenses not covered by Medicare – such as long nursing-home stays) as part of consumption. A household with $h = H$ and consumption $c$ has utility flow

$$u(c) = \frac{[c]^\gamma}{\gamma}.$$  

Following most empirical evidence, let

$$\gamma < 0.$$  

We assume there is a household production technology for transforming expenditure, $x$, to a consumption service flow, $c$:

$$c = \begin{cases} 
  x, & \text{if } h = H \\
  \omega x, & \text{if } h = L 
\end{cases}$$  

(1)
We also assume that the low health state is an impediment to generating consumption services from \( x \); thus,

\[
\omega \in (0, 1).
\]

The loss of consumption services that occurs upon reaching the low health state may be substantial: an agent in need of LTC might lose capacity for home production related to ADLs, and her quality of life may decline precipitously. Utility from consumption expenditure \( x \) while in health state \( h = L \) is

\[
U(x) \equiv u(\omega x) \equiv \omega^n u(x) .
\]

Since \( \omega^n > 1 \), an agent in the low health state has lower utility but higher marginal utility of expenditure. Specifically, marginal utility of consuming \( X \) in low health state equals the marginal utility of consuming a smaller amount, \( X/\Omega \), in high health state:

\[
U'(X) = \frac{\partial u(\omega X)}{\partial X} = \omega u'(\omega X) = u' \left( \frac{X}{\Omega} \right), \text{ where } \Omega = [\omega]^{\frac{1}{1-n}} > 1.
\]

**Available insurance instruments** We assume that state verification problems for \( h \) are much greater than for medical status. An agent knows when he/she enters state \( h = L \), but the transition from \( h = H \) is not legally verifiable. That prevents agents from obtaining health-state insurance.\(^3\) Marshall *et al.* write,

"Indeed, the ultimate luxury good appears to be the ability to retain independence and remain in one’s home ... through the use of (paid) helpers .... These types of expenses are generally not amenable to insurance coverage .... [p.26]

In contrast, all of our model’s households have (Medicare) medical insurance.

Annuities dependent upon the health state are similarly unavailable. In fact, in our baseline case, the analysis treats annuities as exogenously fixed at retirement. However, when discussing the “annuity puzzle,” we allow households to choose their initial portfolio composition. Throughout, we assume that households cannot borrow against their annuities.

**Means-tested public assistance** In our framework, a household with health status \( h = L \) can qualify for Medicaid-provided nursing home care. State verification difficulties affecting private LTC insurance markets may be less relevant for the Medicaid program, because it provides only a basic level of in-kind benefits, and access is rigorously means tested. The means test for this program requires the household to forfeit all of its bequeathable wealth and annuities to qualify for assistance.\(^4\) Let Medicaid nursing home care correspond to expenditure flow \( X_M > 0 \). In practice, elderly households often view Medicaid nursing-home care as a relatively unattractive option.\(^5\) Accordingly, the model incorporates disamenities of Medicaid by assuming that the utility

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\(^3\)On the use of long-term care insurance, which is analogous to health-state insurance in our model, see Miller *et al.* (2010), Brown and Finkelstein (2007, 2008), Brown *et al.* (2012), CBO (2004), and Pauly (1990). Private insurance covers less than 5% of long-term care expenditures in the US (Brown and Finkelstein [2007]). For a discussion of information problems and the long-term care insurance market, see, for example, Norton [2000].

\(^4\)In practice, a household may be able to maintain limited private assets after accepting Medicaid – for example, under some circumstances a recipient can transfer her residence to a sibling or child (see Budish [1995, p. 43]). This paper disregards these program details.

\(^5\)Ameriks *et al.* (2011) refer to disamenities of Medicaid-provided nursing home care as *public care aversion*. Indeed, the level of service is very basic, access is rigorously means tested, and many households strongly prefer to live in familiar surroundings and to maintain a degree of control over their lives (Schafer [1999]).
flow from Medicaid nursing home care is $U(\bar{X})$, where $\bar{X} \leq X_M$ is the expenditure flow adjusted for disamenities.

**Household financial assets** Households retire with endowments of two assets, annuities, with income $a$, and bequeathable net worth $b$ (i.e., liquid wealth). Major components of annuitized wealth include Social Security, defined benefit pension, and Medicare benefits. Bequeathable wealth $b$ pays real interest rate $r > 0$. Let $\beta \geq 0$ be the subjective discount rate. We assume $r \geq \beta$. If we think of the analysis as beginning at age 65, the average interval of $h = H$ might be about 12 years, and the average duration of $h = L$ about 3 years. With a Poisson process, average duration is the reciprocal of the hazard. We assume $A > \lambda > r - \beta$.

**LTC expenditure** Our specification of household preferences assumes the simplest form of state-dependence: utility is $u(x)$ in the high health state and $\omega^\gamma u(x)$ in the low health state, where $x$ is a single consumption category that includes the non-medical part of LTC expenditure. The single-good assumption is not as restrictive as one might think. In fact, a richer model where non-medical LTC expenditure is a separate, endogenous variable would produce an indirect utility function of form (2). To see this, assume that a household has two remaining periods of life and that $h = H$ in the first period and $h = L$ in the last period. Set $r = 0$ and $\beta = 1$; disregard annuities, Medicaid, and uncertain mortality. Then a newly retired household solves

$$\max_x \{u(x) + U(b - x)\}.$$  \hfill (4)

To endogenize the choice of non-medical LTC expenditure, $l$, replace $U(b - x)$ in (4) with

$$U(b - x) \equiv \kappa \cdot \max_l \{\varphi \cdot u(b - x - l) + (1 - \varphi) \cdot u(l)\},$$  \hfill (5)

where $\kappa > 0$ and $\varphi \in (0, 1)$ are preference parameters. Maximization with respect to $l$ in (5) yields exactly the reduced form utility function (2):

$$U(b - x) = \omega^\gamma \cdot u(b - x),$$

$$\omega^\gamma \equiv \kappa \cdot \left([\varphi]^{\frac{1}{\gamma}} + [1 - \varphi]^{\frac{1}{\gamma}}\right).$$

**Non-convexity** Continuing with the two-period example, let Medicaid nursing-home care provide a consumption expenditure flow $\bar{X}$. Accordingly, objective function (4) becomes

$$\max_x \left(u(x) + \max\{U(b - x), U(\bar{X})\}\right).$$  \hfill (6)

Figure 1 depicts the corresponding second-period utility. We can see the non-convexity that Medicaid introduces. Depending on $b$, the optimal solution to (6) is either $x^* = b$ or $x^* < b - \bar{X}$. In other words, the household either consumes all of its wealth while healthy and accepts Medicaid in the

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6 E.g., Sinclair and Smetters [2004].

7 Hubbard et al. [1995] and DeNardi et al. [2010] use a similar specification of preferences but assume that non-medical LTC expenditure is an exogenously fixed parameter not subject to choice, and not directly affecting utility.

8 The two-period example is also convenient for direct comparisons with other two-period models, such as Finkelstein et al. [2013] and Hubbard et al. [1995].

9 This example is similar to the two-period model used in Hubbard et al. [1995].
second period, or it saves enough so that second period consumption exceeds the Medicaid floor $\bar{X}$. It is never optimal to set $x^* \in (b - \bar{X}, b)$.

This introduces complications in a multi-period discrete time framework even if the problem is solved numerically. Furthermore, the non-concave utility function of figure 1 makes a lottery over wealth levels in the appropriate range an attractive way to maximize utility.

Fortunately, by switching to continuous time, our formal model can circumvent both complications above. Doing so allows us to make the age at which liquid wealth is optimally depleted a continuous choice variable (called $T$) separate from expenditure level $x$. Then we can characterize the solution analytically – using standard optimal control methods.

**Summary**

Recapping our baseline assumptions:

**a1:** “Health state” is not verifiable; hence, there is no health-state insurance. Annuities are exogenously set at retirement.

**a2:** If $b_s$ is bequeathable net worth when $h = H$ and $B_s$ is the same for $h = L$, we have $b_s \geq 0$ and $B_s \geq 0$ all $s \geq 0$.

**a3:** $\gamma < 0$, and $\omega \in (0, 1)$.

**a4:** A household transitions from $h = H$ to $h = L$ with Poisson hazard $\lambda$, and from health state $h = L$ to death with Poisson hazard $\Lambda$. We assume $\Lambda > \lambda$.

**a5:** The real interest rate is $r$, with $0 \leq \beta \leq r < \lambda + \beta$.

**a6:** A household in the low health state can turn to Medicaid nursing-home care. The consumption value of the latter is a flow $\bar{X}$.

### 3 Low Health Phase

We solve our model backward, beginning with the last phase of life. In that period, the household is in the low health state $h = L$ and faces mortality hazard $\Lambda$. Without loss of generality, scale the age at which the $h = L$ state begins to $t = 0$. At $t = 0$, let bequeathable net worth be $B \geq 0$. Annuity income is $a \geq 0$, $X_t$ is consumption expenditure at age $t$, and $U(X_t)$ the corresponding utility flow. The expected utility of the household is

$$V(B,a) \equiv \max_{X_t} \int_0^\infty e^{-(\Lambda+\beta)t} U(X_t)dt$$

Below, we show that the household will optimally plan to exhaust its liquid wealth in finite time, which we denote by $T$. If the household is alive at age $T$, it is liquidity constrained and has two options: it can either relinquish its annuity income $a$ and accept Medicaid-provided consumption flow $\bar{X}$, or consume its annuity income for the remainder of its life. Households with $a \geq \bar{X}$ will prefer to live off their annuity income (case (i) below), while households with $a < \bar{X}$ will accept Medicaid assistance (case (ii)). To simplify the exposition, it is convenient to analyze the two cases separately.

**Case (i):** $a \geq \bar{X}$  
Starting from initial wealth level $B$, the household chooses a consumption expenditure path $X_t$ all $t \geq 0$ to solve

$$V(B,a) \equiv \max_{X_t} \int_0^\infty e^{-(\Lambda+\beta)t} U(X_t)dt$$

(7)
subject to \( \dot{B}_t = r \cdot B_t + a - X_t, \)
\[
B_t \geq 0 \quad \text{all} \quad t \geq 0,
\]
\( B_0 = B \) and a given.

The present-value Hamiltonian for (7) is
\[
\mathcal{H} \equiv e^{-(\Lambda + \beta)t}U(X_t) + M_t(rB_t + a - X_t) + N_tB_t,
\]  
with costate \( M_t \), and Lagrange multiplier \( N_t \) for the state-variable constraint \( B_t \geq 0 \). Provided \( M_t \geq 0 \), first-order conditions will be sufficient for optimality provided the transversality condition holds:
\[
\lim_{t \to \infty} M_t \cdot B_t = 0
\]  
(9)

The strict concavity of problem (7) ensures that if a solution exists, it is unique.

We start by formally showing that a household with \( B = 0 \) will optimally set \( X_t = a \) for the remainder of its life.

**Lemma 1:** If \( a \geq \bar{X}, \ (B^*_t, X^*_t) = (0, a) \) is a stationary solution to (7).

**Proof:** See Appendix.

The idea of the proof is that households in (7) behave as if their subjective discount rate is \( \Lambda + \beta > r \); so, a household without a binding liquidity constraint desires a falling time path of consumption expenditure. When \( B_t = 0 \), only \( X_t \leq a \), however, is feasible. At that point, a permanently falling time path cannot be optimal because the household’s liquid wealth would expand until death. Lemma 1 shows that the solution is instead to maintain the constrained outcome forever.

Given Lemma 1, we can construct the general solution to (7) as follows. Suppose the state-variable constraint does not bind until after \( t = T \). Then for \( t \leq T \), omit the term \( N_tB_t \) from the Hamiltonian. The first-order condition for optimal expenditure is
\[
\frac{\partial \mathcal{H}}{\partial X_s} = 0 \iff e^{-(\Lambda + \beta)s} \cdot U′(X_s) = M_s,
\]  
(10)

and the costate equation is
\[
\dot{M}_s = -\frac{\partial \mathcal{H}}{\partial B_s} \iff \dot{M}_s = -r \cdot M_s.
\]  
(11)

Substituting (10) into (11) shows that the optimal expenditure falls at a constant rate:
\[
-(\Lambda + \beta)e^{-(\Lambda + \beta)s}U′(X_s) + e^{-(\Lambda + \beta)s}U''(X_s)\dot{X}_s =
\]
\[
= \dot{M}_s = -r \cdot M_s = -re^{-(\Lambda + \beta)s} \cdot U′(X_s) \iff
\]
\[
(\gamma - 1) \frac{\dot{X}_s}{X_s} = -(r - (\Lambda + \beta)) \iff
\]
\[
\frac{\dot{X}_s}{X_s} = \sigma, \text{ where } \sigma \equiv \frac{r - (\Lambda + \beta)}{1 - \gamma} < 0.
\]  
(12)
Taking into account the household budget constraint, the candidate solutions are depicted on phase diagram figure 2. Each dotted curve is a trajectory satisfying the budget constraint and (12). Equation (12) shows that along each trajectory, \( X_t > 0 \) all \( t \). Nevertheless, we can rule out the optimality of most of the trajectories \textit{a priori}. A given trajectory intersects the vertical line at \( B_0 = B > 0 \) at two points. The higher is preferred. But following the trajectory is then inferior to stopping at the intersection with the line \( X_s = rB_s + a \). Yet the latter cannot be optimal since bequeathable wealth is never exhausted. The exception is the trajectory that intersects the vertical axis at \((0, a)\). Lemma 1 suggests that latter stopping point can be part of an optimal path.

In fact, we can show the transversality condition is then satisfied.

**Proposition 1:** The trajectory in figure 2 that reaches \((B_t, X_t) = (0, a)\) from above and then remains at \((0, a)\) forever solves problem (7). The solution \((B_t^*, X_t^*)\), \( t \geq 0 \), is continuous in \( t \). There exists \( T^* = T^*(B, a) \in [0, \infty) \) such that both \( B_t^* \) and \( X_t^* \) are strictly decreasing in \( t \) for \( t \leq T^* \), but \((B_t^*, X_t^*) = (0, a)\) for \( t > T^* \).

**Proof:** See Appendix.

The next proposition provides additional characterization and establishes solution properties needed for the subsequent phase diagram analysis.

**Proposition 2:** Let \( T^*, B_t^*, \) and \( X_t^* \) be as in Proposition 1. Then \( T^*(B, a) \) is strictly increasing and continuous in \( B \),

\[
T^*(0, a) = 0 \quad \text{and} \quad \lim_{B \to \infty} T^*(B, a) = \infty.
\]

We have

\[
X_t^* = ae^{\sigma(t-T^*)} \quad \text{for} \quad t \in [0, T^*].
\]

As a function of \( B \), \( X_0^* = X_0^*(B, a) \) is continuous, strictly increasing, and strictly concave;

\[
X_0^*(0, a) = a; \quad \text{and} \quad \lim_{B \to \infty} \frac{\partial X_0^*(B, a)}{\partial B} = r - \sigma > 0.
\]

The optimal value function \( V(B, a) \) in (7) is strictly increasing and strictly concave in \( B \).

**Proof:** See Appendix.

**Case (ii):** \( a < \bar{X} \) Case (ii) obtains when the value of Medicaid nursing-home care exceeds a household’s annuity income.

In Lemma 1, a household with \( B = 0 \) chooses \( X_t = a \) forever. In case (ii), the same household could do better by turning to Medicaid. Once Medicaid care is accepted, there is no incentive to ever leave it. In particular, if a household ever exited Medicaid assistance, it would have to start with zero liquid wealth. Subsequent optimal, privately-financed behavior would entail \( X_t = a \) forever. Yet, Medicaid offers, in case (ii), a better alternative, namely, \( X_t = \bar{X} > a \).

Let \( T \) denote the age when the household exhausts its liquid wealth and turns to Medicaid. Then the case (ii) household behavior can be described with a standard free endpoint problem (Kamien and Schwartz [1981, sect.7]):

\[
V(B, a) = \max_{X_t, T} \left( \int_0^T e^{-(\Lambda+\beta)t}U(X_t)dt + e^{-(\Lambda+\beta)T} \frac{U(\bar{X})}{\Lambda + \beta} \right)
\] (13)
subject to $\dot{B}_t = r \cdot B_t + a - X_t$,

$$B_t \geq 0 \quad \text{all} \quad t \geq 0,$$

$$B_0 = B \quad \text{and} \quad a \quad \text{given}.$$

Setting $T = \infty$ in (13) recovers case (i), where accepting Medicaid is never optimal. Note also that formulating problem (13) in continuous time separates the choices of $T$ and $X_t$ and eliminates the non-convexity that would appear if the model were instead cast in discrete time. The following propositions characterize the optimal solution and establish properties necessary for phase diagram analysis.

**Proposition 3:** Problem (13) has a unique solution, $(B^*_t, X^*_t)$, $t \geq 0$. There exists $T^* = T^*(B,a) \in [0, \infty)$ such that both $B^*_t$ and $X^*_t$ are strictly decreasing in $t$ for $t \leq T^*$, but $(B^*_t, X^*_t) = (0, \bar{X})$ for $t > T^*$. $(B^*_t, X^*_t)$ is continuous in $t$ except at $t = T^*$.

Let

$$\bar{X} = \lim_{t \to T^* - 0} X^*_t.$$

There is a unique $\bar{X} = \bar{X}(a) \in (\bar{X}, \infty)$, independent of $B$, such that

$$X^*_t = \begin{cases} \bar{X} \cdot e^{\alpha(t - T^*)}, & \text{for } t \in [0, T^*] \\ \bar{X}, & \text{for } t > T^* \end{cases}.$$

**Proof:** See Appendix.

The analog of Proposition 2 to be used in further analysis is

**Proposition 4:** Let $T^*$, $B^*_t$, and $X^*_t$ be as in Proposition 3. Then $T^*(B,a)$ is strictly increasing and continuous in $B$.

$$T^*(0, a) = 0, \text{ and } \lim_{B \to \infty} T^*(B,a) = \infty.$$

As a function of $B$, $X^*_0 = X^*_0(B,a)$ is continuous (except at $B = 0$) and strictly increasing; we have

$$X^*_0(B,a) = \begin{cases} \text{convex in } B, \quad \left(1 - \frac{a}{\bar{X}}\right) \left(1 - \frac{\alpha}{\bar{X}}\right) > 1 \\ \text{concave in } B, \quad \left(1 - \frac{a}{\bar{X}}\right) \left(1 - \frac{\alpha}{\bar{X}}\right) < (0,1) \end{cases}, \text{ all } B > 0,$$

and,

$$\lim_{B \to \infty} \frac{\partial X^*_0(B,a)}{\partial B} = r - \sigma > 0.$$

The optimal value function $V(B,a)$ in (7) is strictly increasing and strictly concave in $B$.

**Proof:** See Appendix.

**Discussion** A primary difference between cases (i) and (ii) is in the behavior of optimal consumption at age $T^*$ when the liquid wealth is exhausted. Figure 3 illustrates. In case (i), $X^*_t$ continuously approaches its long-run limit $a$. In case (ii), by contrast, optimal consumption jumps down at $t = T^*$.

The discontinuity arises in case (ii) because at age $T^*$, the household exchanges its annuity income flow $a$ for a Medicaid-provided consumption flow $\bar{X} > a$. Consider the household’s trade-offs. Its current optimal consumption at age $T^*$ is $\bar{X}$. Postponing Medicaid for a short time $dt$ forfeits utility $U(\bar{X})dt$. The short-term gain is $U(\bar{X})dt$ less the cost of resources expended.
Annuities represent a sunk cost. The variable private cost is \( [\dot{X} - a]dt \). In utility terms, the cost is \( U'(\dot{X}) \cdot [\dot{X} - a] dt \). The corresponding first-order condition is

\[
U(\dot{X}) dt - U'(\dot{X}) \cdot [\dot{X} - a] dt = U(\dot{X}) dt,
\]

which gives an equation for \( \dot{X} \) as a function of \( \dot{X} \) and \( a \):

\[
U(\dot{X}) - U(\dot{X}) = U'(\dot{X}) \cdot [\dot{X} - a].
\]

(14)

Since the optimal consumption expenditure never drops below \( \dot{X} > a \), in (14) we have \( \dot{X} > a \). Thus, expression (14) implicitly defines an increasing function \( \dot{X}(a) \) to be used in construction of our phase diagrams (see Proposition 5).

The solution methodology illustrates the advantage of our formulation. The lumpiness and means test of Medicaid introduce a non-convexity in a discrete-time formulation – as in figure 1 – making multi-period analysis complicated. Our model, by contrast, circumvents the complications by allowing the household to select the timing of its Medicaid take-up in such a way that it exactly exhausts its liquid wealth first. The discontinuity of \( X^*_t \) in figure 3 might be considered a symptom of the non-convexity of figure 1. Nevertheless, our solution procedure is able to rely upon first-order conditions.

### 4 High Health State Phase

Turn next to households in the healthy phase of their retirement, where \( h = H \). Without loss of generality, rescale household ages to \( s = 0 \) at the start of this phase. A household’s annuity income is \( a > 0 \), and its initial bequeathable net worth (i.e. liquid wealth) is \( b \geq 0 \). With Poisson rate \( \lambda \), the household’s health state changes to \( h = L \), and it receives (recall Section 3) the continuation value \( V(b_s, a) \), where \( b_s \) is its liquid wealth at the time of the transition. Accordingly, a household in state \( h = H \) solves\(^{10}\)

\[
v(b, a) = \max_{x_s} \left( \int_0^\infty e^{-(\lambda+\beta)s} [u(x_s) ds + \lambda V(b_s, a)] ds \right),
\]

\[
\text{s.t. } \dot{b}_s = r \cdot b_s + a - x_s,
\]

\[
b_s \geq 0 \quad \text{all} \quad s \geq 0,
\]

\[
a \geq 0 \quad \text{and} \quad b_0 = b \quad \text{given}.
\]

Concavity of \( V(\cdot) \), shown in the previous section, implies that the integrand in (15) is strictly concave in \((x_s, b_s)\). Analogous to Section 3, the the effective rate of subjective discounting is \( \lambda + \beta > r \).

\(^{10}\)Write the expected utility as

\[
v(b) = \max_{x_s} \left( \int_0^\infty \lambda e^{-\lambda S} \left[ \int_0^S e^{-\beta S} U(x_s) ds + e^{-\beta S} V(b_S) \right] dS \right)
\]

and change the order of integration to obtain (15).
Disregarding the state-variable constraint \( b_s \geq 0 \) for the moment, the present-value Hamiltonian is
\[
\mathcal{H} \equiv e^{-(\lambda+\gamma)t} \cdot [u(x_s) + \lambda \cdot V(b_s)] + m_s \cdot [r \cdot b_s + a - x_s],
\] with \( m_s \) the costate variable. The first-order condition for \( x_s \) is
\[
\frac{\partial \mathcal{H}}{\partial x_s} = 0 \iff e^{-(\lambda+\gamma)s} \cdot u'(x_s) = m_s.
\] (17)
The costate equation is
\[
\dot{m}_s = -\frac{\partial \mathcal{H}}{\partial b_s} = -e^{-(\lambda+\gamma)s} \cdot \lambda \frac{\partial V(b_s)}{\partial b_s} - rm_s.
\] (18)
The law of motion for liquid wealth is
\[
\dot{b}_s = r \cdot b_s + a - x_s.
\] (19)

We construct a phase diagram for \((b_s, x_s)\). Let \( X_0^*(B, a) \) be the initial consumption for the household as it enters the low health state at age \( s \) with liquid wealth \( B = b_s \). The envelope theorem shows that
\[
\frac{\partial V}{\partial B}(B, a) = U'(X_0^*(B, a)).
\] (20)
Eqs (16)-(20) imply
\[
u''(x_s) \cdot \dot{x}_s = -(r - (\lambda + \beta)) \cdot u'(x_s) - \lambda \cdot \omega \gamma \cdot u'(X_0^*(b_s, a)).
\] (21)
Eqs (19) and (21) determine the phase diagram. The isolines of the phase diagram are
\[
\dot{b} = 0 : \quad x = \Gamma_b(b) \equiv r \cdot b + a,
\] (22)
\[
\dot{x} = 0 : \quad x = \Gamma_x(b) \equiv \theta \cdot X_0^*(b, a),
\] (23)
where
\[
\theta \equiv \frac{1}{\Omega} \left[ 1 - \frac{r - \beta}{\lambda} \right]^{-\frac{1}{\delta}} \in (0, 1).
\] (24)
Several distinct phase portraits can arise depending on the shape of \( \Gamma_x(b) \) and the values of exogenous parameters. We begin our analysis of phase diagrams with a lemma that allows us to limit the eventual number of cases.

**Lemma 2:** \( \Gamma_x(b) \) and \( \Gamma_b(b) \) cross at most once.

**Proof:** See Appendix.

Given Lemma 2, the phase portrait of the high health state period depends on the relative magnitudes of \( \Gamma_b(0) \) and \( \Gamma_x(0) \), and on their asymptotic slopes \( \Gamma_b'(\infty) \) and \( \Gamma_x'(\infty) \). Recall that Propositions 1 and 3 imply
\[
\Gamma_b(0) = a, \quad \Gamma_x(0) = \begin{cases} \theta a, & a \geq \bar{X} \\ \theta \bar{X}(a), & a < \bar{X} \end{cases}.
\]
Below we show that there exists \( \bar{a} \in (0, \bar{X}) \) such that
\[
\Gamma_b(0) < \Gamma_x(0) \iff a < \bar{a}.
\] (25)

Turning to the asymptotic slopes of the isoclines, Propositions 2 and 4 and (22) show that
\[
\Gamma_\theta'(\infty) < \Gamma_x'(\infty) \iff r < \bar{r} = \theta (r - \sigma).
\] (26)

It can be shown that inequality (26) will hold when the interest rate is below a threshold (note that \( \theta \) in (24) is also a function of \( r \)). Accordingly, four phase portraits are possible depending on the signs of inequalities (25) and (26). We distinguish between the high annuity case (labelled A) and low annuity case (labelled a) based on the sign of (25). Similarly, the standard interest rate case (labelled r) will obtain when (26) holds, and the high interest rate case (labelled R) will obtain when (26) does not hold. Summarizing, we have

**Proposition 5:** The optimal solution \((x^*_s, b^*_s)\) to (15) is a dotted trajectory on one of the four phase diagrams on figure 4. The phase portrait depends on the parameter values as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>High annuity</th>
<th>Low annuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard interest rate</td>
<td>( r &lt; \bar{r} )</td>
<td>(Ar)</td>
</tr>
<tr>
<td>High interest rate</td>
<td>( r &gt; \bar{r} )</td>
<td>(AR)</td>
</tr>
<tr>
<td>Low annuity</td>
<td>(aR)</td>
<td></td>
</tr>
</tbody>
</table>

where \( \bar{r} \) is defined in (26) and
\[
\bar{a} = \theta \cdot \bar{X} \cdot (1 - \gamma (1 - \theta))^{-\frac{1}{3}}.
\]

**Proof:** See Appendix.

Proposition 5 shows that household total wealth (i.e. liquid wealth plus capitalized annuity income) is not sufficient to predict whether a household in good heath will save or dissave after retirement. It is the level of annuity income, in fact, that plays the pivotal role: \( a \) determines which phase diagram applies regardless of the initial \( b \). We can build further intuitions for figure 4 by examining key trade-offs that shape household optimal behavior.

In figure 4, phase diagrams (Ar) and (aR) have a stationary point at \( b = b^*_\infty = b^*_\infty(a) \). In the former case, we can view \( b^*_\infty(a) \) as a healthy household’s “target level” of liquid wealth: if the household begins retirement with \( b < (>) b^*_\infty(a) \), it will save (dissave) until reaching the target — or falling to health status \( h = L \). In diagram (Ar), on the other hand, \( b^*_\infty(a) \) marks a threshold with respect to Medicaid use: for \( b > b^*_\infty(a) \), a healthy household accumulates wealth to delay its future reliance upon Medicaid; but, if \( b < b^*_\infty(a) \), a household immediately begins dissaving. The following proposition characterizes \( b^*_\infty(a) \) in the case with (Ar).

**Proposition 6** Assume \( r < \bar{r} \) and let
\[
\rho (a) = \frac{b^*_\infty (a)}{a} = \frac{1}{a} \lim_{i \to \infty} b^*_i (b, a)
\]
be the long-run optimal ratio of liquid wealth to annuities. Then
\[
\rho (a) = \begin{cases} 
0, & a \leq \bar{a}, \\
\varsigma (a), & a \in (\bar{a}, \bar{X}), \\
\bar{\rho}, & a \geq \bar{X}.
\end{cases}
\]
where \( \zeta'(a) > 0, \zeta(\bar{a}) = 0, \zeta(\bar{X}) = \bar{\rho}, \) and

\[
\bar{b}^*_t > 0 \iff \frac{b}{a} < \rho(a).
\]

\[(27)\]

**Proof:** See Appendix.

To interpret Proposition 6, one can think of three groups of \( h = H \) households: a low resource group, \( a \leq \bar{a} \); a middle group, \( a \in (\bar{a}, \bar{X}) \); and, a top group, \( a \geq \bar{X} \). In line with this, \( \rho(a) \) has three distinct segments. The bottom segment, \( a \leq \bar{a} \), has \( \rho(a) = 0 \) – the low resource group decumulates wealth starting from any initial level \( b > 0 \). The top segment, \( a \geq \bar{X} \), has a positive and constant \( \rho(a) = \bar{\rho} \), and corresponds to behavior of households that never use Medicaid. For the top group, the desired long-run wealth level is proportionate to the annuity endowment, \( b^*_\infty(a) = \bar{\rho}a \). For the middle group, \( a \in (\bar{a}, \bar{X}) \), the anticipated public benefit discourages liquid wealth accumulation relative to the top group (i.e. \( b^*_\infty(a) < \bar{\rho}a \)). At the same time, the self-insurance motive for the middle group is more sensitive to the annuity income level: \( b^*_\infty(a) = \rho(a)a \) rises more than proportionately with \( a \). The steep rise of \( b^*_\infty(a) \) results from the interaction of the means test and the self-insurance motive: as \( a \) rises, the means test makes the Medicaid benefit less valuable, and this, in turn, strengthens the incentive to self-insure.

### 4.1 Discussion

This subsection provides an intuitive explanation of the phase diagrams of figure 4, which have a key role in the remainder of this paper. Our discussion emphasizes the trade-offs that shape household optimal behavior. As consumption/saving and portfolio choices are closely intertwined, we study them jointly.

**Incomplete markets** Although our model is complicated with, for example, multiple health states, in a first-best environment with symmetric information and complete insurance markets a generalization of Yaari’s [1986] well-known analysis would hold. In particular, a household would optimally rely on state-contingent annuities and insurance contracts, as follows. (i) At retirement, a household would buy an annuity paying a fixed benefit stream for the duration of the high health state. (ii) The household would also buy an insurance policy paying a lump-sum benefit when the high-health state ends. (We refer to this as “long-term care insurance.”) (iii) The household would use the insurance payout to purchase a low-health-state annuity (the return on which would reflect the low-health state mortality rate \( \Lambda \)). A household could complete financial steps (i)-(iii) at the moment of retirement, and it would have no demand for liquid wealth.

Crucially, however, our analysis assumes asymmetric information on each household’s health status. Transactions (i)-(iii) above are then infeasible, and we find that portfolios with a mixture of bonds and annuities provide the highest expected utility.

**Analysis without Medicaid** Consider a household beginning retirement in good health, with annuity income \( a > 0 \), and with (initial) liquid wealth \( b \geq 0 \). For the time being, omit Medicaid (for example, set \( \bar{X} = 0 \)).

In either health status in isolation, there would be no incentive to hold liquid wealth. Specifically, Section 4 (see (15)) shows that during good health, a household effectively has subjective discount rate \( \lambda + \beta \). During poor health, Section 3 (see (7) and (13)) shows the rate is \( \Lambda + \beta \). Assumptions (a4-a5) imply

\[
\Lambda + \beta > \lambda + \beta > r.
\]

\[(28)\]
So, the market return on liquid wealth is less than a household’s internal discount rate. Within a given health regime, a household then has the incentive to decumulate its liquid wealth and attain the corner solution \( b_s = 0 \) and \( x_s = a \) for \( h = H \), or \( B_s = 0 \) and \( X_s = a \) for \( h = L \).

A demand for liquid wealth, nevertheless, can arise from cross-regime differences. Consumption expenditure during good health at rate \( a \) — or higher if the household spends its endowment \( b > 0 \) — leaves marginal utility lower than that which consumption expenditure at rate \( a \) during poor health generates. A resource transfer backward from the \( h = L \) to the \( h = H \) state would violate liquidity constraints; a shift forward, as toward the last phase of life, is, on the other hand, feasible. In the absence of Medicaid, optimal behavior dictates the latter.

During good health, a household will then husband at least a part of its initial liquid wealth \( b \) — or even add to it by saving a portion of its annuity income. After \( h = H \) ends, the analysis of Section 3 begins, and the household spends down its liquid wealth within a finite time. During the spend-down, the household’s consumption expenditure, say, \( X_s \), exceeds \( a \). The interval with \( X_s > a \) constitutes a household’s reward for carrying liquid wealth to the \( h = L \) state. After the liquid wealth’s exhaustion, the household’s consumption expenditure settles permanently to \( X_s = a \).

**Analysis Including Medicaid**

Next, reintroduce Medicaid. Continue with the household above. Now, \( \bar{X} > 0 \). Given the latter, derive \( \bar{a} \) as in Proposition 5. Begin with the case \( r < \bar{r} \).

We interpret \( \bar{a} \) as follows. When \( a < \bar{a} \), Medicaid is sufficiently generous relative to the household’s private standard of living that the household does not wish to transfer resources to the \( h = L \) state. In view of (28), the household then systematically spends, even during good health, its initial liquid wealth \( b \). And, it accepts Medicaid care promptly once \( h = L \). Figure 4 (ar) illustrates.

When, on the other hand, \( a \in (\bar{a}, \bar{X}) \), the household’s private resources are great enough to make the standard of living under Medicaid care unappealing by comparison. Then the household strives, during good health, to reach a target liquid wealth \( b^*_{\infty}(a) > 0 \) — see Propositions 5-6. It husbands the latter until \( h = L \). After \( h = L \), it uses the liquid wealth to postpone Medicaid take-up (perhaps delaying past its survival date). This is the case that figure 4 (Ar) illustrates.

If \( a \geq \bar{X} \), Medicaid is irrelevant to a household (recall Section 3); so, the analysis of the preceding subsection applies. In terms of Propositions 5-6, we can see that \( a > \bar{X} \) implies \( a > \bar{a} \). For \( r < \bar{r} \), we again have \( b^*_{\infty}(a) > 0 \). Phase diagram (Ar) continues to hold.

When \( r > \bar{r} \), the analysis is largely the same — though liquid wealth becomes even more attractive. If, for instance, \( a > \bar{a} \), we have \( b^*_{\infty}(a) = \infty \). (See figure 4 (AR).) If \( a < \bar{a} \), there exists \( b^*_{\infty}(a) > 0 \) such that \( b > b^*_{\infty}(a) \) inspires a household to save throughout good health to delay reliance upon Medicaid. (See figure 4 (aAR).)

**Magnitude of \( b^*_{\infty}(a) \)**

In the case \( r < \bar{r} \) and \( a > \bar{a} \), the magnitude of the liquid wealth target \( b^*_{\infty}(a) \) determines whether early in retirement a household saves (which it does when \( b < b^*_{\infty}(a) \)) or dissaves (which it does when \( b > b^*_{\infty}(a) \)). Proposition 6 shows that \( b^*_{\infty}(a) \) is increasing in \( a \), and that the increase is rapid — faster than linear — for the middle class. These properties become especially important in the next section. Their intuition is as follows.

The model is homothetic in \((b, a, \bar{X})\). So, \( b^*_{\infty}(a) \) is linearly homogeneous in the tuple. Since \( b^*_{\infty}(a) \) is independent of \( b \), we then have

\[
b^*_{\infty}(k \cdot a, k \cdot \bar{X}) = k \cdot b^*_{\infty}(a, \bar{X})
\]

(29)

A middle-class household’s financial gain from accepting Medicaid is \( \bar{X} - a \) (recall that Medicaid confiscates a household’s annuity income). In (29), the gain is \( k \cdot \bar{X} - k \cdot a \). If we multiply \( a \), but not
\(\bar{X}\), by \(k\), on the other hand, the gain from Medicaid, \(\bar{X} - k \cdot a\), is smaller, creating more incentive to accumulate liquid wealth. Hence,

\[
b_\infty^* (k \cdot a, \bar{X}) > b_\infty^* (k \cdot a, k \cdot \bar{X}) = k \cdot b_\infty^* (a, \bar{X}) \text{ for } k \geq 1.
\]

In other words, \(b_\infty^* = b_\infty^* (a)\) — with \(\bar{X}\) implicitly held constant — rises faster than linearly in \(a\).

### 4.2 Summary

This paper elaborates a standard life-cycle model to include means-tested Medicaid nursing-home care; several health states, with different marginal utilities of consumption expenditure and different Poisson hazards; and, asymmetry of information on the health status of individual households. The new elements play a substantial role in the outcomes in Sections 3-4. The new elements are, in particular, important determinants of the model’s phase diagrams.

We find that the option for Medicaid nursing-home care divides households with annuity incomes \(a < \bar{X}\) into 2 groups. Those with \(a \in (\bar{a}, \bar{X})\) are ambivalent about Medicaid: they employ it a back-up because it is free, but they dislike the low standard of living it provides and, to delay reliance upon it, they husband and/or accumulate liquid wealth while their health status is high. Those with \(a < \bar{a}\) are less averse to living on Medicaid and, accordingly, have no inclination to hold significant balances of liquid wealth.

The model shows that household life-cycle portfolio and consumption/saving choices mirror one another. On the one hand, limited financial market options force households into “second-best” optimal behavior. Resulting incentives for self-insurance lead, for example, to different wealth trajectories in different segments of the earnings distribution, as described above. Conversely, as middle-class households attempt to prepare for end-of-life needs, they demand liquid wealth — so that both the size and the composition of portfolios end up reflecting household (dynamic) resource allocation plans.

### 5 Saving after retirement

Although the standard life-cycle model implies that households will systematically dissave late in life, survey data often seems to show cohort post-retirement average liquid wealth changing only slowly with age, perhaps even increasing. The Introduction refers to this inconsistency as the “retirement saving puzzle.” The present section suggests that as we enhance our modeling framework with Medicaid, multiple health states, and asymmetries of health information, the discrepancy between theory and evidence will tend to diminish appreciably.

Section 4 shows that healthy households may continue to save after retirement, or at least, may want to husband their existing liquid wealth. Here, we demonstrate that healthy households can remain a significant fraction of cohort survivors long after retirement. Combining the two results, we then show that a cohort’s average liquid wealth need not decline with age.

**Post-retirement saving** Section 4 finds that some households may, while their health status remains favorable, want to continue accumulating wealth after retirement. Initial conditions, in particular, a household’s annuity income, are an important factor.

Proposition 5 partitions households into 3 groups. We have a low-resource group, \(a \leq \bar{a}\); a middle-class group, \(a \in (\bar{a}, \bar{X})\); and, a top group, \(a \geq \bar{X}\). Households in the low-resource group
tend to spend their liquid wealth promptly, beginning during good health. They then subsist on their annuity income until poor health makes them eligible for Medicaid, which they find relatively attractive. The middle-class group, in contrast, builds a nest egg of liquid wealth \( b_\infty^*(a) > 0 \). The target nest egg is increasing in \( a \). (In fact, Proposition 6 shows that even the ratio \( b_\infty^*(a)/a \) is increasing in \( a \).) If a household in this category begins retirement with liquid wealth \( b < b_\infty^*(a) \), it saves until \( b = b_\infty^*(a) \) or \( h = L \). After the onset of poor health, it spends its liquid wealth and, after the latter is gone, accepts Medicaid. The \( a \geq \bar{X} \) group also has a liquid-wealth target during good health. In poor health, after spending down the liquid wealth, these households live on their annuity income.

The richness of the set of possible behaviors hints that the model may be able to rationalize otherwise paradoxical post-retirement outcomes. We now examine that possibility further.

**Cohort Composition** The evidence on post-retirement wealth that has attracted the most attention measures average (liquid) wealth, at different ages, for an individual birth cohort’s survivors. Fortunately, our model allows a detailed description of cohort wealth trajectories. We begin with an examination of the evolution of a cohort’s mixture of health states.

Consider a cohort of retired, single-person households. In the model, all begin retirement with health status \( h = H \). Each subsequently transitions to \( h = L \), then to death. As the households age, the cohort size steadily diminishes. Somewhat paradoxically, however, the ratio of survivors in high versus low health converges to a positive constant.

To see this, let \( f_H(t) \) be the fraction of households remaining alive and in good health \( t \) years after retirement. Then

\[
f_H(t) = e^{-\lambda t}.
\]

Similarly, let the fraction alive at \( t \) but in low health status be

\[
f_L(t) = \int_0^t \lambda e^{-\lambda s} e^{-\Lambda(t-s)} ds.
\]

Combining expressions, the fraction of survivors in high health status is

\[
f(t) = \frac{f_H(t)}{f_H(t) + f_L(t)} = \frac{1}{1 + \frac{\Lambda}{\Lambda - \lambda} \cdot (1 - e^{-(\Lambda - \lambda) t})}.
\]

Provided \( \Lambda > \lambda \), \( f(t) \) falls monotonically from \( f(0) = 1 \) to \( f(\infty) = (\Lambda - \lambda)/\Lambda > 0 \). With \( \lambda = 1/12 \) and \( \Lambda = 1/3 \) (recall the illustration in Section 2), for instance, \( f(\infty) = 3/4 \).

Although our Poisson processes may only be approximations, they illustrate that healthy households can comprise a substantial fraction of cohort survivors long into retirement. This is important because, as noted above, retirees in good health can behave quite differently from those whose health is poor.

**Cohort Average Wealth** We now can develop a full characterization of a specific cohort’s long-run average liquid wealth. For the short run, simulations illustrate that many outcomes are possible — including, as we shall see, outcomes resembling those in the data.

**Long-Run Outcomes.** Begin with a cohort of single-person, healthy households each with the same endowment \((b, a)_R\). Normalize the cohort size to 1.

---

11 This description is somewhat over-simplified if \( r > \bar{r} \) — see case \((aR)\) in figure 4.
Using the notation of Sections 3-4, a household remaining in high health status \( t \) periods after retirement has liquid wealth \( b_t^* = b_t^*(b, a) \). The total wealth of a cohort of agents who remain healthy is

\[
b_H(t; b, a) = e^{-\lambda t} \cdot b_t^*(b, a).
\]

The wealth of cohort survivors in the low health state depends on the age at which their health status changed. If a household enters low health status \( s \leq t \) periods after retirement, its initial wealth upon entering that state is \( B = b_s^*(b, a) \). The household subsequently follows the low-health-status optimal wealth trajectory (recall Section 3). At time \( t \), it has passed \( t - s \) years in low health status, and its wealth is \( B_t^*(B, a) \). The fraction of a cohort entering the low health state at age \( s \) and surviving until age \( t \) is \( \lambda \cdot e^{-\lambda s} \cdot e^{-\lambda(t-s)} \). Accordingly, the total wealth of agents who are in low health \( t \) periods into retirement is

\[
b_L(t; b, a) = \int_0^t \lambda e^{-\lambda s} e^{-\Lambda(t-s)} B_t^*(b_s^*(b, a), a) \, ds.
\]

Cohort average wealth is total wealth divided by the number of survivors:

\[
\bar{b}(t; b, a) = \frac{b_H(t; b, a) + b_L(t; b, a)}{f_H(t) + f_L(t)}.
\]

An analytic characterization for \( \bar{b}^*(a) \equiv \lim_{t \to \infty} \bar{b}(t; b, a) \) is possible.

**Corollary to Proposition 5** The long-run cohort average wealth, \( \bar{b}^*(a) \), depends on exogenous parameters as follows:

<table>
<thead>
<tr>
<th>Standard interest rate</th>
<th>High annuity ( a &gt; \bar{a} )</th>
<th>Low annuity ( a &lt; \bar{a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r &lt; \bar{r} )</td>
<td>( b^<em>(a) \in b^</em>_\infty(a) \cdot \left[ \frac{\Lambda - \lambda}{\Lambda}, 1 \right] )</td>
<td>( b^*(a) = 0 )</td>
</tr>
<tr>
<td>( r &gt; \bar{r} )</td>
<td>( \bar{b}^*(a) \to \infty )</td>
<td>( \bar{b}^<em>(a) = \left{ \begin{array}{ll} 0, &amp; b &lt; b^</em><em>\infty(a) \ \infty, &amp; b &gt; b^*</em>\infty(a) \end{array} \right. )</td>
</tr>
</tbody>
</table>

The proof is straightforward. The cases in which \( \bar{b}^*(a) \) is zero or infinity follow directly from Proposition 5 and figure 4. The case in which \( \bar{b}^*(a) \) is positive and finite corresponds to phase diagram (Ar). The bounds are intuitive. New entrants to the low health group have (liquid) wealth no greater than \( b^*_\infty(a) \); consequently, members of the \( h = L \) group have wealth that is non-negative but bounded above by \( b^*_\infty(a) \). The long-run contribution of the group to cohort average liquid wealth is between 0 and \( (1 - f(\infty)) \cdot b^*_\infty(a) \). The wealth of the high-health-status group converges to \( f(\infty) \cdot b^*_\infty(a) \). The sum of the two contributions, \( \bar{b}^*(a) \), therefore must lie in the interval

\[
b^*_\infty(a) \cdot [f(\infty), 1] = b^*_\infty(a) \cdot \left[ \frac{\Lambda - \lambda}{\Lambda}, 1 \right].
\]

This establishes the Corollary.

If \( r < \bar{r} \), we can see that in the long run, a cohort with some high annuity households should have positive stationary average liquid wealth. For \( r > \bar{r} \), long-run average liquid wealth should diverge to \( \infty \). The possibility of a level, or rising, cohort wealth trajectory depends on the asymptotic
stationarity of (30) and on Section 4’s finding that healthy retirees may husband their wealth or continue to accumulate more.

Simulated wealth trajectories: narrow wealth ranges. We utilize numerical simulations in illustrating our model’s ability to match empirical outcomes in the short run. We consider 2 comparisons.

Section 2 suggests parameter values \( \lambda = 1/12, \Lambda = 1/3, \text{ and } \bar{X} = \xi \cdot X^M \) for \( \xi \in (0, 1] \). Appendix 2 calibrates \( \Omega \). Appendix 2 also suggests cross-sectional quantiles for \( a \) — see Table A1 — and notes values for \( \gamma, r, \text{ and } \beta \) familiar from the literature. Table A2 determines corresponding phase diagrams for the model.

We first compare post-retirement cohort trajectories of average (liquid) wealth for the model with empirical profiles from DeNardi et al. [2015, fn.4]. We simulate age-wealth profiles for the model for selected parameters within the Appendix-2 domain. DeNardi et al. derive graphs of cohort wealth from HRS/AHEAD panel data on single-person households aged 74 or older in 1996.\(^{12}\) Convenient features of the empirical graphs are that they segregate the underlying sample into narrow annuity-income bands (i.e., into quintiles of the cross-sectional distribution of \( a \)) and that, because the median age of retirement in the US is about 62, even the youngest households in the graphs have often been retired for over a decade. The latter implies that the ratio of health types may well have virtually completed its convergence to \( f(1) \) in (30).\(^{13}\) A complication, on the other hand, is that the number of data points is fairly small, and becomes increasing so at higher ages (c.f., DeNardi et al. [2015, fn.4]). The asymptotic stationarity of our ratio \( f(t) \) depends on large samples. Accordingly, we ignore the jagged regions at the right-hand ends of the empirical graphs.

Figure 5 presents illustrative simulations from the model. The simulations assume \( f(t) = f(\infty), r = \beta = 0.02, \text{ and } \bar{X} = 52.5 \), and they consider \( \gamma = -0.75, -1.0, \text{ and } -1.25 \). In all cases, \( r < \bar{\tau} \).

For \( a \) below the median of Table A1, Table A2 then implies phase diagram \( (ar) \) — i.e., \( a < \bar{a} \) — with prompt spend-down of liquid wealth regardless of health. That behavior is consistent with the low and declining wealth balances evident in the 2 bottom-quintile empirical graphs. The model provides an intuitive explanation, namely, that low-annuity households do not perceive that they can do better, once stricken with poor health status, than to depend upon Medicaid. For \( a \) near the (Table-A1) median, similar parameter values yield \( b \approx b^*_{\infty}(a) > 0 \) in Table A2. Hence, by age 74, the corresponding simulated wealth trajectory is nearly horizontal. Again, that seems broadly consistent with the empirical graphs. Finally, for simulations of the top 30 and top 10% annuity groups, Table A2 implies much higher values of \( b^*_{\infty}(a) \) — as Proposition 6 would predict. In particular, \( b^*_{\infty}(a) \) tends to be large relative to \( b \), leading to simulated age-wealth trajectories that rise for a number of years. The top-quintile graph of the data rises from age 74 to 84-86 at a rate of 0.5-1.5%/yr. For the simulations, the average wealth of the top 30 and 10% of households rises 0.5 to 1.7%/yr. Intuitively, high-annuity households demand high liquid wealth balances to reduce their future reliance upon Medicaid.

Figure 5 suggests that, for plausible parameter values, simulations from the model can match empirical trajectory shapes, and that, through Propositions 5-6, our theoretical analysis can provide explanations for the behavior arising in practice.

Simulated wealth trajectories: broad population averages

Second, we compare the model with empirical figures from Poterba et al. [2011]. Single graphs from the latter summarize a full cross-section

---

\(^{12}\)DeNardi et al.’s panel data can avoid birth-cohort fixed effects and complications from correlations of survival probability with portfolio size.

\(^{13}\)With \( \lambda = 1/12 \) and \( \Lambda = 1/3 \), for example, convergence to \( f(\infty) \) is over 98% complete after 12 years.
of annuity incomes. And, the data tend to begin at the empirical retirement age, so that the convergence of (30) runs its course as we move along a graph. Nonetheless, this has been an important form for evidence in the literature, and we can again use our model to interpret the data’s patterns.

As above, Poterba et al. use panel data. They link average liquid wealth holdings in adjoining survey waves, including only households with data in both waves. They process the data extensively, using trimmed means and medians. We focus on the graphs of Poterba et al. [2011, fig. 1.10 & 1.11], which combine 5 age groups. These graphs include only single households. But, as noted below, they are not limited to retirees.

We can compare simulated median liquid wealth with Poterba et al. [2011, fig. 1.11]. Roughly speaking, the empirical graph is horizontal for households in their late 60s, and falls -0.6%/yr for households in their 70s. Medians may be less sensitive to non-retirees than means. Our comparison group from the model is healthy households with median initial conditions (i.e., \((b, a) = (21, 100)\)).

For simplicity, we use only single age groups in this case. We use the same parameters as above. Thus, we have \(r < \bar{r}\). And, for a median household, \(a > \bar{a}\). The phase diagram is \((\text{Ar})\). Outcomes are straightforward, as follows: for \(b < (>) \hat{b}_\infty(a)\), the liquid wealth of healthy households monotonically rises (falls), until becoming stationary at the target level, \(b_\infty(a)\). For \(\gamma = -1.25, -1.0, \text{ and } -0.75\), the 15-year growth rates of simulated liquid wealth are, respectively, 1.4%/yr, 0.7%/yr, and -0.04%/yr. The last is evidently the best match.\(^{14}\)

Figure 5 (right panel) simulates cohort mean wealth trajectories from the model. The simulations use Table-A1 endowments \((b, a) = (14, 15), (100, 21), (272, 34)\) and \((57, 892)\), with weights 1/3, 1/3, 2/9 and 1/9, respectively. Parameter values continue to be as in the preceding subsection. For conformity with Poterba et al. [2011, fig. 1.10] 5-year age groups, the graphs in figure 5 present 5-year moving averages. The empirical graphs reveal a growth rate of about 1.3%/yr for 5 years, and 1.4%/yr for the next 10. The simulated curves show a brief dip, from the large initial (percentage) increases in low-health status households (the total peak-to-trough being 0.7%, 2.3%, and 5.0% for \(\gamma = -1.25, -1.0, \text{ and } -0.75\), respectively). Thereafter, the curves grow at rates 1.1%/yr, 0.9%/yr, and 0.6%/yr for \(\gamma = -1.25, -1.0, \text{ and } -0.75\), respectively. The presence of non-retired households in the data may explain part of the remaining discrepancies.

Thus, even in the most challenging case, Poterba et al. [2011, fig 1.10], the illustrative simulations can match the data quite well. Our qualitative analysis shows why. If \(r < \bar{r}\), poor health or very low annuity income lead to declining liquid wealth (see Proposition 5). Households with moderate annuity income accumulate wealth more slowly than high annuity, healthy households (see Figure 5 and Proposition 6). We show the rising and falling segments can counterbalance one another in the weighted average, and the time-varying cohort composition can flatten the initial portion of the average trajectory.

Discussion For decades, evidence of the “retirement-saving puzzle” has raised questions about the validity of the life-cycle model. We, however, argue that several elaborations of the standard framework, which are interesting and realistic in their own right, can greatly improve the model’s performance. The enhanced model’s ability to match the evidence includes both aggregative data and data on separate income groups.

Our analysis uses both qualitative results and straightforward numerical simulations. The latter utilize plausible parameters values. The former enable us to shed light on the possible causes of otherwise surprising cohort wealth trajectory patterns evident in practice.

\(^{14}\)For \(\gamma = -0.50\), the simulated (15-year) growth rate would drop to -1.1%/yr.
6 Demand for annuities

Standard life-cycle theories addressing self-insurance of longevity risk – starting with well-known work of Yaari [1965] – have been hard to reconcile with households’ apparent lack of demand for annuities at retirement. Using our model, we now reconsider this “annuity puzzle.”

To study demand for annuities, this section deviates from our baseline specification to allow households to re-allocate their portfolios at retirement. Let

$$r_A = \frac{(\lambda + r) (A + r)}{\lambda + A + r}$$

(33)
denote the actuarially fair rate of return used to capitalize annuity income (see the Appendix for the derivation of (33)). Then household total initial wealth, $w_0$, can be expressed through its endowment of liquid wealth and annuities $(b_0, a_0)$ prior to portfolio choice, as follows:

$$w_0 = b_0 + \frac{a_0}{r_A}.$$ 

At retirement, a household re-allocates its endowed total wealth between bonds and annuities to maximize its post-retirement value function. The resulting optimal allocation $(b, a)$ becomes the initial state for the household optimization problem (15). To formulate the household portfolio choice problem, it is convenient to define

$$\alpha_0 = \frac{a_0/r_A}{w_0} = \frac{a_0}{a_0 + r_A b_0},$$

the initial share of annuitized wealth at retirement. Then the household problem is

$$\hat{\alpha}(w_0) = \arg \max_{\alpha \in [0,1]} v((1 - \alpha) w_0, r_A \alpha w_0).$$

(34)

If the desired annuity share, $\hat{\alpha}$, exceeds the endowed share $\alpha_0$, a household will exhibit demand for annuities at retirement.

To develop intuitions, first consider the role of annuities for low-resource households, that is, the group following phase diagram (ar) or phase diagram (aR) with $b < b^*_c$. Medicaid provides better support in low health state than such households could otherwise afford; therefore, they are content to accept Medicaid promptly after poor health begins. Annuities provide insurance against outliving one’s resources during good health; Medicaid provides longevity protection once $h = L$. But, Medicaid usurps a household’s annuity income, causing annuities to lose part of their appeal. The expected present value of an annuity stream $a = A \cdot r_A$ useful only during good health is

$$a \cdot \int_0^\infty e^{-(\lambda + r)s} ds = A \cdot r_A \frac{A + \lambda}{\lambda + A + r} < A.$$

With $\lambda = 1/12$ and $A = 1/3$ and $r = 0.02$ (0.03), for example, we have

$$\frac{A + \lambda}{\lambda + A + r} = 0.81 (0.81) < 1.$$ 

In other words, an actuarially fair annuity carries, roughly speaking, an inherent user cost (or “load”), which is likely to be non-trivial.
The inherent cost can be even larger for middle class households with \( a \in (\bar{a}, X) \). The one-size-fits-all Medicaid benefit \( X \) leaves them dissatisfied. Hence, they carry resources to the low-health state to postpone reliance upon Medicaid. To do so, a middle-class household augments its annuities with bonds. Upon Medicaid take-up, a household must relinquish its annuities and remaining bonds to the public authority. As in Section 3, a household can consume both the income and principal of its bonds prior to accepting Medicaid. Unlike bonds, however, annuities are illiquid. Recall that utility in state \( h = L \) is

\[
\int_0^\infty e^{-(\beta + \Lambda)t} \cdot U(X_t) dt.
\]

Since \( \Lambda \) tends to be large, even if bond-wealth is used up rather quickly after the onset of poor health, total utility can significantly benefit. Relying exclusively on accumulating bonds during good health is risky, as the good health phase may turn out to be brief. Starting with a mixture of annuities — to protect against a long span of \( h = H \) — and bonds — to delay the need to accept Medicaid if the span of \( h = H \) turns out to be short — becomes attractive.

Put another way, purchasing an annuity income \( a \) at retirement has a price \( A = a/r_A \). When the low health state arrives, the actuarially fair capitalized value of the same income flow drops to \( a/(\Lambda + r) \). The capital loss can be substantial: the value of \( a \) after \( h = L \) as a fraction of its purchase price is

\[
\frac{a}{a/r_A} = \frac{r_A}{\Lambda + r} = \frac{\lambda + r}{\lambda + \Lambda + r}.
\]

Letting \( \Lambda = 1/3 \) and \( \lambda = 1/12 \), for instance, the relative value in (35) is 0.24 (0.25) when \( r = 0.02 \) (0.03) — a roughly 75 percent capital loss. If the household subsequently turns to Medicaid, it must relinquish \( a \) to the Medicaid program. At that moment, the value to the household of the annuity income declines further, to 0. These are steep drops. What is more, their timing is extremely inopportune: at the onset of \( h = L \), a household’s marginal-utility-of-consumption function rises abruptly. And, Proposition 3 shows that as a household accepts Medicaid, its consumption (discontinuously) drops. Evidently, annuities subject a household to severe capital losses exactly at times when the household values consumption highly. Bond values, in contrast, are unrelated to health. At Medicaid take-up, a household essentially must hand over its remaining bonds. But, as Section 3 shows, households can spend their bond wealth completely prior to that moment. Roughly speaking, in the last stage of life annuities and Medicaid are substitutes, whereas bonds and Medicaid are complements.

The arguments above do not apply to very high annuity households, that is to say, those with \( a \geq X \). The latter households never use Medicaid. Their total wealth must exceed \( \bar{w} \) with

\[
\bar{w} \geq \frac{a}{r_A} \geq \frac{X}{r_A}.
\]

With \( \lambda \) and \( \Lambda \) as above and \( r = 0.02 \) (0.03),

\[
\frac{X}{r_A} = 11.96 \cdot X \ (10.85 \cdot X).
\]

In Table A2, only top-decile households have \( w \geq \bar{w} \). **Illustrative examples** We present solutions to the household portfolio choice problem for different initial wealth levels to illustrate the impact of Medicaid availability on the demand for annuities
at retirement. Table 1 shows the wealth components, the initial share of annuitized wealth and the solutions to (34) for the 30th, 50th, 70th and 90th percentiles of the empirical wealth distribution of Table A1. The exogenous parameters are set to $r = \beta = 0.02$, $\gamma = -1$, $\lambda = 1/12$, $\Lambda = 1/3$, $\bar{X} = 52.5$ and $\Omega = 5.25$, consistent with figure 5.

$$a_0 \ b_0 \ w_0 \ \alpha_0 = \frac{a_0}{a_0 + r A b_0} \ \hat{\alpha}\big|_{\bar{X}=0} \ \hat{\alpha}$$

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<tbody>
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<td>0.58</td>
<td>0.93</td>
<td>0.48</td>
</tr>
<tr>
<td>57</td>
<td>892</td>
<td>1571</td>
<td>0.43</td>
<td>0.93</td>
<td>0.93</td>
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</table>

Table 1. Portfolios at retirement: actual, optimal (no Medicaid), optimal (with Medicaid)

For comparison, consider the case without Medicaid first, i.e., set $\bar{X} = 0$. Proposition 5 shows that $\bar{X} = 0$ implies $a > \bar{a} = 0$; hence, phase diagram (Ar) applies. Without Medicaid long-term care, the model is homothetic in $(b, a)$, and the optimal share of annuitized wealth at retirement, $\hat{\alpha}\big|_{\bar{X}=0}$, is independent of household total wealth. Evidently, absent Medicaid, the model exhibits the annuity puzzle in rows 2-4: the desired share of annuitized wealth, $\hat{\alpha}\big|_{\bar{X}=0}$, exceeds the initial share, $\alpha_0$ in all rows except the first.\(^{15}\)

The last column reports the optimal share of annuitized wealth, $\hat{\alpha}$, setting $\bar{X} = 52.5$. For rows 1-3, the annuity puzzle has disappeared: in rows 1-2, actual annuitization is equal or slightly larger than desired; and, in row 3, the actual is 20% above desired. Our model provides an interpretation. As in Section 5, we have $r < \bar{r}$. According to the model, row-1 households, with $a < \bar{a}$, will find standard annuities attractive. And, Table A1 shows they are heavily annuitized in practice as well. In the middle class, with $a \in (\bar{a}, \bar{X})$, the model implies households will desire mixed portfolios, using liquid assets to postpone reliance on Medicaid.

Annuity-puzzle behavior does emerge in row 4 of Table 1. One possibility is that a somewhat higher choice of $\xi$ would make $a < \bar{X} = \xi X_M$ for the top group (recall that $X_M = 70$ and $a = 57$). Another is that the millionaires in the top decile want to leave intentional bequests – behavior which is outside the scope of our modeling.

The analysis suggests a possible resolution of the annuity puzzle, at least for households with middle-class annuity incomes: the limited annuitization that households have in practice may accurately reflect their preferences, given the availability of Medicaid long-term care and the treatment of annuity income in the means test.

7 Medicaid Take-up and Bequests

The basic assumptions of our model enable it to offer interpretations of interesting phenomena in addition to the retirement-saving and annuity puzzles. This Section briefly describes 2 examples. The timing of Medicaid take-up DeNardi et al. [2013] presents evidence that even households with relatively high annuity income sometimes use Medicaid nursing-home assistance very late in

\[^{15}\text{Our framework differs from Yaari [1965] in that mortality hazard is correlated with (state-dependent) marginal utility, and this explains why households desire less than 100\% annuitization. However, the deviation from full annuitization is slight — an outcome that is reminiscent of other recent analyses, e.g. Davidoff et al. [2005].}\]
life, though households with lower $a$ tend to access Medicaid more frequently and at younger ages. Our model offers an intuitive explanation for these outcomes.

Proposition 3 shows that any household with $a < \bar{a}$ will access Medicaid if it survives long enough. The model determines Medicaid take-up time as a function of a retiree’s initial condition $(b, a)$ and age at the onset of poor health. If $S$ is the time spent in good health, then the optimal age of Medicaid take-up is $S + T^*(b_\ell^*(b, a), a)$, where the function $T^*(\cdot)$ is as in Section 3. The model thus provides a mapping between portfolio composition at retirement, household health history, and Medicaid take-up age — making a comprehensive treatment possible.

Consider the low interest rate case. Households with $a < \bar{a}$ want to accept Medicaid promptly once $h = L$. Households with $a > \bar{a}$, on the other hand, hold bonds to postpone their resort to Medicaid. These households are more likely to die before Medicaid take-up, or to begin Medicaid only at advanced ages.

**Bequest behavior** Households in the model leave accidental bequests if they die before spending down their liquid wealth. Survey questions suggest that such bequests may be important in practice, while evidence on intentional bequests has been mixed.\(^{16}\)

In our model, a household begins health state $h = L$ with liquid wealth $B \geq 0$. It spends the latter at a rapid rate (with its consumption flow exceeding $\bar{X}$). If it dies before exhausting $B$, the residual constitutes a bequest. If it lives longer, it has no bequest, and it finishes life relying upon Medicaid or its annuity income, whichever is larger.

Proposition 5 determines interest-rate and annuity thresholds, $\bar{r}$ and $\bar{a}$. Consider first a low-resource household, i.e. one following phase diagram $(ar)$ or phase diagram $(aR)$ with $b < b_\ell^\ast$. Section 4 shows the household will dissave as long as it remains in the good health state. If $h = H$ lasts long enough, it will begin poor health with liquid wealth $B = 0$. Since all households dissave in poor health, the household would then die with no estate. If $B > 0$, it would subsequently decumulate its liquid wealth rapidly, leaving a bequest only if it died before $B_\ell^\ast$ reached 0. In general, households in the low-resource group will tend to leave estates only if their life span in both segments of retirement is short.

Alternatively, suppose a household has (i) $r > \bar{r}$, $b > b_\ell^\ast$, and $a < \bar{a}$; (ii) $r > \bar{r}$ and $a > \bar{a}$; (iii) $r < \bar{r}$, $a > \bar{a}$, and $b < b_\ell^\ast$; or, (iv) $r < \bar{r}$, $a > \bar{a}$, and $b > b_\ell^\ast$. In case (iv), the household dissaves during good health with a lower limit $b_\ell^\ast$. In the remaining 3 cases, it saves while $h = H$. It begins $h = L$ with liquid wealth $B > 0$. It fully dissaves its liquid wealth after (finite) time span $T^*(B, a)$ (recall Section 3). The function $T^*(\cdot)$ is increasing in $B$ and decreasing in $a$. The household leaves an estate if it dies within $T < T^*(B, a)$ years. In cases (i)-(iii), a longer time spent in good health leads to a higher probability of leaving an estate. In cases (i)-(iv), a shorter life span after $h = L$ always makes a positive estate more likely. For the same $B$, a higher annuity income $a$ makes a bequest less likely. Our propositions offer a full characterization of the timing and magnitude of such transfers.

8 Conclusion

This paper presents a life-cycle model of post-retirement household behavior emphasizing the roles of changing health status (correlated with changes in mortality), annuitized wealth, and Medicaid

\(^{16}\)E.g., Altonji, Hayashi, and Kotlikoff [1992, 1997], Laitner and Ohlsson [2001], and others.
assistance with long-term care. Despite the presence of health-status uncertainty and the nonconvexity introduced by the Medicaid means test, our analysis yields a deterministic optimal control problem where the solution can be characterized with phase diagrams.

Qualitatively (and quantitatively in calibrated examples), we show the model is consistent with the gently rising cohort post-retirement wealth trajectories that tend to appear in data. Similarly, we show that a sizeable fraction of households may not wish to buy additional annuities at retirement — with both Medicaid LTC and existing primary annuitization from Social Security and DB pensions playing important roles in the outcome. The model can, in other words, offer a unified explanation for two long-standing empirical puzzles, the “retirement-saving puzzle” and the “annuity puzzle.”

The model shows that after retirement but while in good health, middle-class households may want to maintain, or continue to build, their non-annuitized net worth. Households value primary annuities for the income that they provide, bonds for flexibility of access to funds, and Medicaid LTC for backstop protection against extreme longevity. Primary annuities and bonds can assume complementary roles: middle-class households may, during good health, save part of their annuity income to (temporarily) support a higher living standard in poor health than Medicaid nursing-home care provides. In the model, this behavior can be understood to be a consequence of state-dependent utility and incomplete financial and insurance markets.
References


[16] DeNardi, Mariacristina; Eric French; and John Bailey Jones, “Medicaid Insurance in Old Age,” NBER working paper 19151, June 2013.


Appendix 1 Calibration and numerical results

**Calibration** Our model has a limited number of parameters. We set \( \lambda = 0.0833 \) and \( \Lambda = 0.3333 \), corresponding to time intervals of 12 and 3 years, respectively, as in Sinclair and Smetters [2004]. The literature has a variety of estimates of \( \gamma \leq 0 \) (see, for example, Laitner and Silverman [2012]) and generally uses \( \beta \in [0, 0.04] \). We consider values \( \gamma \in [-0.5, -3.0] \), corresponding to a coefficient of relative risk aversion \( 1 - \gamma \in [1.5, 4] \), and values \( r, \beta \in [0.02, 0.03] \).

The model includes two parameters that are less familiar: \( \Omega \) – defined in (3) – which captures the rise in marginal utility associated with the low health state, and \( \bar{X} \), which measures the value to a recipient household of Medicaid nursing-home care.

The proposed calibration exploits the fact that Medicaid is a social-insurance program. Theoretically, \( \bar{X} \) might be thought of as a choice variable for a social planner who seeks to insure the target recipient of public long-term care. Accordingly, a comparison of with the normal expenditure of a healthy target recipient identifies the difference in marginal utility across states that would rationalize \( \bar{X} \).

Think of the target recipient as a household that would quickly turn to Medicaid upon reaching the low health state, and let \( \bar{x} \) denote the recipient’s expenditure level while still healthy. Efficiency requires equalizing marginal utilities of expenditure across health states:

\[
U'(\bar{X}) = u' (\bar{x}). \tag{36}
\]

In the model, households that are quick to accept Medicaid enter the low health state, say, at age \( s \), with nearly zero liquid wealth, \( b_s = B \approx 0 \) (see phase diagram (ar)). Since \( b_s \approx 0 \), the typical recipient’s consumption just prior to \( s \) must be \( \bar{x} \approx a \), so that \( U'(\bar{X}) = u' (a) \) in (36). Optimality condition (36) then relates \( \bar{X} \) and \( \Omega \) as follows:

\[
\Omega = \frac{\bar{X}}{a}. \tag{37}
\]

Condition (37) enables us to use data on Medicaid nursing-home reimbursement amounts and target-recipient annuity incomes to evaluate \( \Omega \). To calibrate \( a \), we assume, as above, that a target Medicaid recipient has low initial liquid wealth and an annuity income substantially below the population median. We set \( a = \bar{a} = 10,000 \), which is about one-half of population median, and about 2/3 of the annuity income of the 30th percentile among single-person retired households (see Table A1, column 4)\(^{17}\).

To estimate the effective long-term care consumption flow \( \bar{X} \), we start with a direct measure of nursing home care cost, \( X_M \). In Met Life [2009], annual average expenditures for nursing-home care in 2008 are $69,715 for a semi-private room, and $77,380 for a private room.\(^{18}\) Accordingly, we set \( X_M = 70,000 \). Prior studies (e.g., Ameriks et al. [2011], Schafer [1999]) suggest that \( \bar{X} \) might be much lower than \( X_M \). Reasons might include the disutility of living in an institution and/or accepting government welfare. Accordingly, for a fixed \( X_M = 70,000 \), let

\(^{17}\)By way of comparison, the chosen value of \( a = 10,000 \) is somewhat higher than the annual SSI amount (7,644 2008 dollars) that acts as a lower bound on household annuity income in practice. All else equal, calibrating from a lower \( a \) would produce a higher \( \Omega \) and supply a stronger self-insurance motive. We prefer calibrations of \( \Omega \) on the low side to stack the cards against the post-retirement saving behavior that our model is trying to explain.

\[
X = \xi \cdot X_M, \quad \xi \leq 1.
\]

We report results for \(\xi \in \{0.5, 0.75, 1\}\), which imply \(X \in \{35000, 52500, 70000\}\) and \(\Omega = \{3.5, 5.25, 7.0\}\). The resulting middle estimate, \(X = 52500\), is close to the calibrated consumption floor in nursing-home eligible state in Ameriks et al. [2011] – their estimate of \(X\) is 56,300 (2008 dollars).

**Numerical results** Table A2 provides calculations that illustrate Proposition 5 and the qualitative results of Section 5. Each panel of the table corresponds to a distinct vector of exogenous parameters \((r, \beta, \Omega, X)\) consistent with (37) and reports the values of \(a\) and \(b^*_\infty\) for a set \(\gamma \in \{-0.5, -1, -2, -3\}\).

We can see that all four phase diagrams of figure 4 obtain for empirically relevant parameter values. To illustrate the model’s predictions, we take several initial conditions \((b, a)\) from the balance sheets of single-person households aged 65-69 reported in Poterba et al. [2011, Table 2]. Table A1 shows the corresponding components of annuitized and non-annuitized wealth at selected points of the wealth distribution.\(^{19}\)

Consider a household at the 30-th percentile of the annuitized wealth distribution in Table A1 with \(a = 15\) and \(b = 14\). Table A2, columns 4 and 5, show the phase diagram types and the values of \(b^*_\infty\) corresponding to \(a = 15\). For all \(\gamma > -3\) (CRRA less than 4), the model predicts that households with \(a = 15\) and \(b = 14\) should dissave, either because they follow phase diagram \((ar)\) or because they follow phase diagram \((ar)\) and have a low initial wealth \(b = 14 < b^*_\infty\).

Next, take a household with a median annuity income \(a = 21\) and the corresponding liquid wealth \(b = 100\). Table 2, columns 6 and 7 show that the model’s predictions with respect to wealth accumulation depend on the risk aversion parameter. When risk aversion is low (e.g. \(\gamma = -0.5\)), the phase diagram type is \((ar)\) with \(b > b^*_\infty\), where Proposition 6 would imply wealth decumulation. As risk aversion rises, so does \(b^*_\infty\). Accordingly, for higher levels of risk aversion (\(\gamma \leq -1\)), we have \(b < b^*_\infty\), and the model predicts post-retirement saving.

The above logic extends to behavior of households all the way to the top of the wealth distribution. For instance, take a household with \(a = 34\) and \(b = 272\) corresponding to the 70-th percentile. Table A2, column 9 shows that \(b^*_\infty\) is large, as \(b^*_\infty(a)\) in Proposition 6 rises rapidly with \(a\). The model then predicts \(b < b^*_\infty\), at least for \(\gamma < -0.5\). At higher levels of risk aversion (CRRA 3 or 4), column 8 shows that phase diagram switches to \((AR)\), where the model predicts wealth accumulation starting from arbitrarily large \(b\).

Broadly, the patterns of Table A2 seem consistent with observations on wealth accumulation behavior of single-person households showing post-retirement saving at higher wealth levels and flat or falling wealth at lower wealth levels (e.g. Poterba et al. [2012], De Nardi et al. [2015]).

Our analysis stresses portfolio composition at retirement as an important determinant of post-retirement saving. It is therefore worth explaining why Table A1 data might show households at retirement with annuity-heavy portfolios. If agents anticipate a need to save after retirement, then why did not they save more before retirement? We think that one answer has to do with composition of single-person households by marital status. According to the US Census [2012], 42 percent of single-person households aged 65-74 are widowed, and an additional 40 percent are divorced. Thus the Table A1 wealth distribution used as the initial condition for the model describes mostly single households who experienced a past shock to family status. Both divorce and death

\(^{19}\)Poterba et al. [2011] use the actuarially fair rate of return on annuities to capitalize annuity flows. Consistent with this, we use the actuarially fair rate of return \(r_A\) from (33) to convert between annuity wealth and income flow.
of a spouse deplete wealth: Poterba et al. [2011, Figures 2, 4] show a sharp drop in non-annuity financial assets following a transition from two- to one-person household. By contrast, married couples and continuing singles show a rising wealth-age profile. In line with this, the data show that single-person households are more heavily annuitized than couples –70 percent annuitization for a median single household versus 57 percent for a median married couple (Poterba et al. [2011, Table 2]).

Appendix 2 Proofs

Proof of Lemma 1. Suppose \((B_t, X_t) = (0, a)\) all \(t\). Consider Hamiltonian (8). The first-order condition for \(X_t\) yields
\[
e^{-\lambda \lambda t} U'(a) = M_t. \tag{38}
\]
The costate equation yields, after substituting from (38),
\[
\dot{M}_t = -r M_t - N_t \Leftrightarrow \]
\[
-(\lambda + \beta)e^{-\lambda \lambda t} U'(a) = -r e^{-\lambda \lambda t} U'(a) - N_t \Leftrightarrow \]
\[
N_t = -[r - (\lambda + \beta)]e^{-\lambda \lambda t} U'(a). \tag{39}
\]
By assumption, \(-[r - (\lambda + \beta)] > 0\). So, \(N_t \geq 0\). And, the time-path of the Lagrange multiplier is continuous. \(B_t = 0\) all \(t\) in this lemma. Hence, \(N_t \cdot B_t = 0\). Similarly, we can see that transversality condition (9) holds.

Proof of Proposition 1 Refer to Hamiltonian (8). Let \((B^*_t, X^*_t)\) be the trajectory that converges to \((0, a)\) from above. Equation (12) shows the vertical motion in figure 2 is strictly negative. Let \(T^* < \infty\) be the time \((B^*_t, X^*_t)\) reaches \((0, a)\). For \(t \leq T^*\), the budget constraint of (7) together with (12) determine the shape of \((B^*_t, X^*_t)\); (10) determines \(M_t\). Set \(N_t = 0\).

For \(t > T^*\), set \(N_t, M_t, X^*_t\), and \(B^*_t\) as in the proof of Lemma 1. Then the first-order condition for \(X_t\), the costate equation, the budget equation, and the state-variable constraint all hold for \(t \geq 0\); we have \(N_t \geq 0\) all \(t\); the path of \(N_t\) is piecewise continuous; \(N_t \cdot B_t = 0\) all \(t\) by construction; the costate variable is non-negative all \(t\) and continuous by construction; and, transversality condition (9) holds. Hence, \((B^*_t, X^*_t)\) is optimal.

Proof of Proposition 2 Expression (12) shows
\[
X^*_t = X^*_0 \cdot e^{\sigma T^*}. \tag{40}
\]
By construction, \(X^*_T = a\). So,
\[
X^*_0 = a \cdot e^{-\sigma T^*}. \tag{39}
\]
Budget accounting then implies
\[
B = \int_0^{T^*} e^{-rt} (a \cdot e^{-\sigma (T^* - t)} - a) \, dt, \tag{40}
\]
which determines \(T^* = T^*(B, a)\). From (40), we can see that \(T^*(B, a)\) is a strictly increasing and continuous function of \(B\), with
\[
\lim_{B \to \infty} T^*(B, a) = \infty, \tag{41}
\]
and
\[
T^*(0, a) = 0. \tag{42}
\]

Turning to the properties of \(X^*_0(B, a)\), we can then see from (39) that \(X^*_0(B, a)\) is continuous and strictly increasing in \(B\); (42) implies \(X^*_0(0, a) = a\).

Differentiating (40) with respect to \(B\) gives
\[
\frac{\partial T^*}{\partial B} = \frac{1}{-a \cdot \sigma \cdot e^{-\sigma T^*} \cdot \int_0^{T^*} e^{(\sigma-r)t} dt}.
\]

Differentiating (39),
\[
\frac{\partial X^*_0}{\partial B} = -\sigma \cdot a \cdot e^{-\sigma T^*} \cdot \frac{\partial T^*}{\partial B}.
\]

Combining the last two expressions,
\[
\frac{\partial X^*_0}{\partial B} = \frac{1}{\int_0^{T^*} e^{(\sigma-r)t} dt}.
\]
Since \(T^*(B, a)\) is increasing in \(B\), (43) implies \(X^*_0\) is concave in \(B\). Given (41), (43) also establishes
\[
\lim_{B \to \infty} \frac{\partial X^*_0(B, a)}{\partial B} = r - \sigma.
\]

Finally, the envelope theorem shows
\[
\frac{\partial V}{\partial B}(B, a) = U''(X^*_0(B, a)).
\]
Hence, \(V(B, a)\) is continuously differentiable; and, because \(X^*_0\) is strictly increasing in \(B\), \(V(B, a)\) is strictly concave in \(B\).

**Proof of Proposition 3**

Step 1. Fix \(a\) and \(\bar{X}\). In case (ii), we have \(a < \bar{X}\). Define a function
\[
\pi(X) \equiv U(X) - U(\bar{X}) + U'(X) \cdot (a - X), \quad \text{all } X > a. \tag{44}
\]
This function is continuous and strictly increasing in \(X\), and it has opposite signs at the ends of the interval \([\bar{X}, \infty)\):
\[
\pi'(X) = U''(X) \cdot (a - X) > 0,
\]
\[
\pi(\bar{X}) = U'(\bar{X}) \cdot (a - \bar{X}) < 0,
\]
\[
\lim_{X \to \infty} \pi(X) = -U(\bar{X}) + \lim_{X \to \infty} \left( U(X) + \gamma \cdot U(X) \cdot \frac{a - X}{X} \right) = -U(\bar{X}) > 0.
\]

34
It follows that on $\bar{X}, \infty$, $\pi(X)$ has a unique root. Denote this root $\bar{X}$.

Step 2. Optimality requires that once $B_t = 0$, the household permanently accepts Medicaid. Prior to that, the Hamiltonian is (8) with $N_t = 0$. The first-order condition, costate equation, and budget equation are as in case (i). Hence, the phase diagram is as in figure 2. For any initial $B \geq 0$, choose the trajectory at the top of the diagram that converges to $(0, \bar{X})$. As in case (i), convergence takes a finite time (which we denote $T^*$). Assume Medicaid take-up for $t > T^*$, with $X_t^* = \bar{X}$.

The concavity of the problem and the discussion in the text show that we know optimal behavior conditional on $T^*$. Let

$$W(T) = e^{-(\Lambda+\beta)T} \frac{U(\bar{X})}{\Lambda+\beta}$$

denote the continuation value of accepting Medicaid at date $T$. Kamien and Schwartz [1981, p. 143] show that the first-order conditions for the optimal acceptance date $T^*$ are

$$B_{T^*} \geq 0, \quad M_{T^*} \geq \frac{\partial W(T^*)}{\partial B_{T^*}} \geq 0, \quad B_{T^*} \cdot \left[ M_{T^*} - \frac{\partial W(T^*)}{\partial B_{T^*}} \right] = 0,$$

$$\mathcal{H}_{t-T^*} + \frac{\partial W(T^*)}{\partial T} = 0,$$  \hspace{1cm} (46)

where we use the Hamiltonian from (8) without the state-variable constraint. Our proposed solution has

$$B_{T^*} = 0.$$  \hspace{1cm} (47)

From (10), $M_{T^*} > 0$. $W(T)$ is not a function of $B_{T^*}$, making its partial derivative 0. Hence, our proposed solution is consistent with (45). Evaluating (46) at $T = T^*$ yields

$$\pi(X_{T^*}) \cdot e^{-(\Lambda+\beta)T^*} = 0.$$  \hspace{1cm} (48)

Hence, Step 1 establishes (46).

By construction, we have $\bar{X} = \lim_{t \to T^* - 0} X_t^*$ and

$$X_t^* = \left\{ \begin{array}{ll} \bar{X} \cdot e^{\sigma(t-T^*)}, & \text{for } t \in [0, T^*] \\ \bar{X}, & \text{for } t > T^* \end{array} \right.$$

It remains to show that the first-order condition for $T^*$ is sufficient. We have argued that the root of $\pi(\cdot)$ is unique. Suppose we choose a larger (smaller) $T^*$. The trajectories of figure 2 remain as before. Thus budgetary accounting implies we must lower (raise) $\bar{X}$ for our stationary point accordingly, leading to $\pi(\bar{X}) < (>)0$. Hence, the right-hand side of first-order condition (46) yields a maximum at our original $T^*$. $\blacksquare$

**Proof of Proposition 4.** The proof follows that of Proposition 2. We concentrate here on the convexity/concavity of $X_0^*(B, a)$ and its asymptotic behavior.

As in the proof of Proposition 2, $X_0^* = X_{T^*}^* \cdot e^{-\sigma T^*}$. In this case, $X_{T^*}^* = \bar{X}$. So,

$$X_0^* = \bar{X} \cdot e^{-\sigma T^*}$$  \hspace{1cm} (49)

and

$$\frac{\partial X_0^*}{\partial B} = -\sigma \cdot \bar{X} \cdot e^{-\sigma T^*} \cdot \frac{\partial T^*}{\partial B}.$$  \hspace{1cm} (50)
Budgetary accounting implies

\[ B = \int_0^{T^*} e^{-rt} \cdot (X_0^* (B, a) e^{\sigma t} - a) \, dt. \]

Differentiating the above with respect to \( B \) yields

\[ 1 = e^{-rT^*} \left( X_0^* \cdot e^{\sigma T^*} - a \right) \frac{\partial T^*}{\partial B} + \frac{\partial X_0^*}{\partial B} J(T^*), \]

where \( J(T) \equiv \int_0^T e^{-(r-\sigma)t} \, dt \) (50)

Substituting from (48)-(49) into (50), we have

\[ \frac{\partial X_0^*}{\partial B} = 1 \frac{D(T^*)}{D(T^* (B, a))} \]

where \( D(T^*) = -\frac{1}{\sigma} \cdot \frac{\dot{X} - a}{X} \cdot e^{-(r-\sigma)T^*} + J(T^*). \)

The asymptotic behavior of \( \frac{\partial X_0^*}{\partial B} \) follows:

\[ \lim_{B \to \infty} D(T^*(B, a)) = \lim_{T^* \to \infty} D(T^*) = \lim_{T^* \to \infty} J(T^*) = \frac{1}{r - \sigma}. \]

The convexity or concavity of \( X_0^*(B, a) \) follows as well:

\[ D'(T^*) = \frac{r - \sigma}{\sigma} \cdot \frac{\dot{X} - a}{X} \cdot e^{-(r-\sigma)T^*} + e^{-(r-\sigma)T^*} = \left[ 1 - \left( 1 - \frac{a}{X} \right) \left( 1 - \frac{r}{\sigma} \right) \right] \cdot e^{-(r-\sigma)T^*.} \]

**Proof of Lemma 2.** Suppose \((b^*, x^*)\) is a solution to (22)-(23) for a fixed \( a \) and \( \bar{X} \).

Step 1. Suppose \( a \geq \bar{X} \).

Proposition 4 and (23) imply

\[ x^* = \theta \cdot X_0^*(b, a) = \theta \cdot \bar{X} \cdot e^{-\sigma T} \]

where \( T = T^*(b^*, a) \). Let \( Z = \bar{X}/a \). Then the equation for \( b^* \) reads

\[ \theta aZe^{-\sigma T} = rb^* + a \iff \frac{b^*}{a} = \frac{1}{r} \left[ \theta Ze^{-\sigma T} - 1 \right]. \] (51)

As in the proof of Proposition 4, budgetary accounting yields

\[ b^* = \int_0^T e^{-rt} \left( aZe^{\sigma t} - a e^{-rT} \right) \, dt \iff \]

\[ \frac{b^*}{a} = Z e^{-\sigma T} - e^{-rT} \frac{r - \sigma}{r} - \frac{1 - e^{-rT}}{r} \] (52)

Equating \( \frac{b^*}{a} \) in (51)-(52), we have

\[ \frac{1}{r} \left( \theta Ze^{-\sigma T} - 1 \right) = Z e^{-\sigma T} - e^{-rT} \frac{r - \sigma}{r} - \frac{1 - e^{-rT}}{r} \iff \]

\[ e^{-rT} \left( \frac{Z}{r - \sigma} - \frac{1}{r} \right) = Z e^{-\sigma T} \left( \frac{1}{r - \sigma} - \frac{\theta}{r} \right) \iff \]
\[ e^{(r - \sigma) T Z \left( \frac{1}{r - \sigma} - \frac{\theta}{r} \right)} = \left( \frac{Z}{r - \sigma} - \frac{1}{r} \right) \]  

(53)

The last expression depends on \( b^* \) only through \( T \). (53) either has a unique solution \( T > 0 \) or no solution. If \( T > 0 \) exists, Proposition 4 shows that \( T = T^*(b, a) \) is strictly increasing in \( b \); hence \( b^* \) must be unique if \( T \) is unique.

**Step 2**. If \( a < \bar{X} \), repeat Step 1 argument setting \( Z = 1 - \) recall Proposition 2. ■

**Proof of Proposition 5**

**Step 1** Suppose \( a \geq \bar{X} \). Then

\[ a \geq \bar{X} > \theta \cdot \bar{X} \cdot (1 - \gamma (1 - \theta))^{-\frac{1}{\gamma}} = \bar{a}, \]

and

\[ \Gamma_b(0) = a > \theta X^*(0, a) = \theta a = \Gamma_x(0), \]

so we have left-hand side diagrams on figure 4. The asymptotic slope from Proposition 2 establishes the cases that obtain for \( r < (<) \theta \cdot (r - \sigma) \).

**Step 2** Suppose \( a < \bar{X} \). We show that there exists a unique \( \bar{a} \in (0, \bar{X}) \) such that

\[ \Gamma_b(0) = a < \theta \bar{X}(a) = \Gamma_x(0) \Leftrightarrow a < \bar{a}. \]

Consider \( \pi(.) \) from (44) and make a change of variables

\[ \bar{X}(a) = a Z(a). \]  

(54)

Since \( \pi(\bar{X}(a)) = 0 \) implies that \( \bar{X}(a) > a \), we have \( Z(a) > 1 \). Using (2) and (54), equation \( \pi = 0 \) can be written as

\[ (1 - \gamma) Z^\gamma + \gamma Z^{\gamma - 1} = \left( \frac{a}{\bar{X}} \right)^{-\gamma} \]  

(55)

The left-hand side of (55) is strictly decreasing in \( Z \) for all \( Z \geq 1 \), with

\[ Z(\bar{X}) = 1, \quad \lim_{a \to 0} Z(a) = \infty \text{ and } Z'(a) < 0. \]

Hence, there is a unique \( \bar{a} \in (0, \bar{X}) \) with

\[ \bar{a} = \theta \bar{X}(\bar{a}) \Leftrightarrow Z(\bar{a}) = \frac{1}{\theta} \]  

(56)

Evaluating (55) at \( Z = 1/\theta \) and \( a = \bar{a} \) gives

\[ \bar{a} = \bar{X} \cdot \theta (1 - \gamma (1 - \theta))^{-\frac{1}{\gamma}}. \]

Since \( Z(a) \) is strictly decreasing,

\[ a < \theta \bar{X}(a) \Leftrightarrow \frac{1}{\theta} < Z(a) \Leftrightarrow a < \bar{a}. \]

**Step 3** Step 2 shows that \( a > (<) \bar{a} \) separates the left and right-hand side diagrams in figure 4. Lemma 2 and the asymptotic slopes in Proposition 4 complete the proof. ■
Proof of Proposition 6 Suppose $\theta \cdot (r - \sigma) > r$. Let $\bar{X}(a) = aZ(a)$ as in (54). Define $Z(a)$ for all $a \geq \bar{a}$ as follows:

$$Z(a) = \begin{cases} \frac{1}{\bar{a}}\bar{X}(a), & \text{if } a \in [\bar{a}, \bar{X}) \\ 1, & \text{if } a \geq \bar{X} \end{cases}.$$  

Then Proposition 5 shows $Z(a)$ is continuous for all $a \geq \bar{a}$ and strictly decreasing for $a \in [\bar{a}, \bar{X})$. From (56) we have

$$1 \leq Z(a) \leq \frac{1}{\bar{a}}. \quad (57)$$

In the proof of Lemma 2, (51) shows

$$\frac{b^*}{a} = \frac{\theta Z(a)e^{-\sigma T^*} - 1}{r}. \quad (58)$$

And, (53) relates $T^*$ and $Z$:

$$e^{(r-\sigma)T^*} = \frac{1}{r} - \frac{Z}{r-\sigma}.$$  

In the low interest rate case, $\theta \cdot (r - \sigma) > r$ and (57) imply that both the numerator and the denominator of the above expression are positive. Define

$$\psi(Z) \equiv Z e^{-\sigma T^*} = Z \left[ \frac{1}{Z_T} - \frac{1}{r-\sigma} \right]^{-\frac{\sigma}{r-\sigma}}.$$  

Then, from (58)

$$\frac{d}{da} \left( \frac{b^*}{a} \right) = \frac{\theta}{r} \psi'(Z) \cdot Z'(a).$$

Showing that $\psi'(Z) < 0$ for all $Z \in (1, 1/\theta]$ and $\psi'(1) = 0$ would complete the proof. Indeed,

$$\frac{d}{dZ} \ln \psi(Z) = \frac{1}{Z} + \sigma \frac{1}{r-\sigma} - \frac{1}{rZ} = \frac{1}{Z} \frac{Z-1}{rZ-1} < 0.$$  

The numerator of the above expression is negative for all $Z > 1$ and zero for $Z = 0$. The denominator is positive when $r < \bar{r}$ and $Z < 1/\theta$. ♦  

Derivation of the actuarially fair rate of return on annuities Let $A$ be the market value of an annuity with income $a$. Then

$$a = Ar,$$  

(59)

If $\mathbb{E}_T[.]$ is the expectation over the stochastic life-span $\bar{T}$, we have

$$A = \mathbb{E}_T \left[ \int_0^{\bar{T}} e^{-rt} \, dt \right] = a \int_0^{\infty} \lambda e^{-\lambda t} \int_0^T e^{-rt} \, dt \, dT + \int_0^{\infty} \lambda e^{-\lambda T} \int_T^{\infty} \Lambda e^{-\Lambda s} \, ds \, ds dT.$$  

(60)

The first right-hand side term registers annuity income during the healthy phase of retirement, the second term gives income during the last phase of life. Performing the integration and combining (59)-(60), we have

$$r = \frac{\lambda + r}{\lambda + \Lambda + r}.$$  

38
Figure 1. Second period utility for optimization problem (6)

Figure 2. Trajectories obeying first-order conditions and the budget constraint, but not the transversality condition.

Figure 3. Optimal consumption path in low health state. Case (i): $a \geq \bar{X}$, Case (ii): $a < \bar{X}$. 

Figure 4. Possible phase diagrams for optimal behavior in high health state. See Proposition 5.

Figure 5. Cohort average wealth trajectories by initial wealth level, $\lambda = \frac{1}{12}, \Lambda = \frac{1}{3}, r = \beta = 0.02, \Omega = 5.25, \bar{X} = 52.5.$
### Appendix Tables

<table>
<thead>
<tr>
<th>Net Worth</th>
<th>Annuitized Wealth</th>
<th>Annuity Income, $a_0$</th>
<th>Bequeathable Wealth, $b_0$</th>
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<tr>
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<td>Soc. Security, $b_{SS}$</td>
<td>DB Pension, $b_{DB}$</td>
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</tr>
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<td>0</td>
<td>15</td>
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<tr>
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</tr>
<tr>
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<td>34</td>
</tr>
<tr>
<td>90&lt;sup&gt;th&lt;/sup&gt; Percentile</td>
<td>388</td>
<td>292</td>
<td>57</td>
</tr>
</tbody>
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Table A1. Primary annuities and bequeathable wealth (000s of 2008 dollars) for single-person households Aged 65-69.

Source: Poterba <i>et al.</i> [2011], Table 2 and p. 99.

<table>
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<th>$\gamma$</th>
<th>$\bar{a}$</th>
<th>$\bar{r}$</th>
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<td>$b_{\infty}$</td>
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<td>Panel 1</td>
<td>$r = 0.02, \beta = 0.02, \Omega = 7.0, \bar{X} = 70$</td>
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Table A2. Phase Diagram Types for Various Parameter Combinations (see Figure 4 and Proposition 5).

Fixed parameters: $\Lambda = 1/3, \lambda = 1/12$. 